

EXISTENCE OF A MINIMAL NON-SCATTERING SOLUTION TO THE MASS-SUBCRITICAL GENERALIZED KORTEWEG-DE VRIES EQUATION

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ABSTRACT. In this article, we prove existence of a non-scattering solution, which is minimal in some sense, to the mass-subcritical generalized Korteweg-de Vries (gKdV) equation in the scale critical \hat{L}^r space where $\hat{L}^r = \{f \in \mathcal{S}'(\mathbb{R}) \mid \|f\|_{\hat{L}^r} = \|\hat{f}\|_{L^{r'}} < \infty\}$. We construct this solution by a concentration compactness argument. Then, key ingredients are a linear profile decomposition result adopted to \hat{L}^r -framework and approximation of solutions to the gKdV equation which involves rapid linear oscillation by means of solutions to the nonlinear Schrödinger equation.

1. INTRODUCTION

In this article, we consider generalized Korteweg-de Vries (gKdV) equation

$$(gKdV) \quad \begin{cases} \partial_t u + \partial_x^3 u = \mu \partial_x (|u|^{2\alpha} u), & t, x \in \mathbb{R}, \\ u(0, x) = u_0(x) \in \hat{L}^\alpha(\mathbb{R}), & x \in \mathbb{R} \end{cases}$$

where $u : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is an unknown function, $u_0 : \mathbb{R} \rightarrow \mathbb{R}$ is a given data, and $\mu = \pm 1$ and $\alpha > 0$ are constants. The space \hat{L}^r is defined for $1 \leq r \leq \infty$ by

$$\hat{L}^r = \hat{L}^r(\mathbb{R}) := \{f \in \mathcal{S}'(\mathbb{R}) \mid \|f\|_{\hat{L}^r} = \|\hat{f}\|_{L^{r'}} < \infty\},$$

where $\hat{f} = \mathcal{F}f$ stands for Fourier transform of f with respect to space variable and $r' = (1 - 1/r)^{-1}$ denotes the Hölder conjugate of r with conventions $1' = \infty$ and $\infty' = 1$. We call that (gKdV) is defocusing if $\mu = +1$ and focusing if $\mu = -1$. Our aim here is to study time global behavior of solutions to (gKdV) with focusing nonlinearities in the *mass-subcritical* range $\alpha < 2$. More specifically, we investigate existence of a threshold solution which lies on the boundary of small scattering solutions around zero and other solutions.

The class of equations (gKdV) arises in several fields of physics. Equation (gKdV) is a generalization of the Korteweg-de Vries equation which models long waves propagating in a channel [32]. Equation (gKdV) with $\alpha = 1$ is also known as the modified Korteweg-de Vries equation which describes a time evolution for the curvature of certain types of helical space curves [33].

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The equation (gKdV) has the following scale property; if $u(t, x)$ is a solution to (gKdV), then

$$u_\lambda(t, x) := \lambda^{\frac{1}{\alpha}} u(\lambda^3 t, \lambda x)$$

is also a solution to (gKdV) with a initial data $u_\lambda(0, x) = \lambda^{\frac{1}{\alpha}} u_0(\lambda x)$ for any $\lambda > 0$. When $\alpha = 2$, (gKdV) is called mass-critical because the above scale leaves the mass invariant.

The small data global existence results of (gKdV) in scale critical spaces have been studied by several authors. Kenig-Ponce-Vega [27] proved the small data global well-posedness and scattering of (gKdV) in the scale critical space \dot{H}^{s_α} for $\alpha \geq 2$, where $s_\alpha := 1/2 - 1/\alpha$ is a scale critical exponent. Since the scale critical exponent s_α is negative in the mass-subcritical case $\alpha < 2$, well-posedness of (gKdV) in \dot{H}^{s_α} becomes rather a difficult problem. Tao [54] proved global well-posedness for small data for (gKdV) with the quartic nonlinearity $\mu \partial_x(u^4)$ in $\dot{H}^{s_{3/2}}$. Later on, Koch-Marzuola [31] simplified Tao's proof and extended his result to a Besov space $\dot{B}_{\infty, 2}^{s_{3/2}}$. As for the \hat{L}^r -framework, Grünrock and his collaborator proved well-posedness for various nonlinear dispersive equations, see [15, 16, 17].

On the other hand, the asymptotic behavior in time of solution to (gKdV) is studied for the small initial data in weighted Sobolev spaces [53, 52, 47, 11, 18, 19, 20]. It is known that $\alpha = 1$ is a critical exponent for scattering problem of (gKdV). More precisely, if $\alpha > 1$, then solution to (gKdV) converges to solution to Airy equation $\partial_t v + \partial_x^3 v = 0$ (see [18]) and if $0 < \alpha \leq 1$, then solution to (gKdV) does not converge to solution to Airy equation (see [48, 21]). Furthermore, for the case $\alpha = 1$, Hayashi-Naumkin [19, 20] proved a existence of modified scattering states for (gKdV). Note that for the case $\alpha = 1$, (gKdV) is completely integrable and the inverse scattering method is available. By using the inverse scattering method, Deift-Zhou [12] obtained more precise asymptotic behavior in time of solution to (gKdV) with $\alpha = 1$.

The well-posedness of (gKdV) and small data scattering in \hat{L}^α is established by the authors as long as $8/5 < \alpha < 10/3$ by introducing a generalized version of Stichartz's estimates adopted to the \hat{L}^r -framework, see [42]. The mass

$$M[u] = \frac{1}{2} \|u\|_{L^2}^2$$

and the energy

$$E[u] = \frac{1}{2} \|\partial_x u\|_{L^2}^2 + \frac{\mu}{2\alpha + 2} \|u\|_{L^{2\alpha+2}}^{2\alpha+2}$$

are well-known conserved quantities for (gKdV). However, neither makes sense in general for \hat{L}^α -solutions. Thus, global existence is nontrivial for large data even in the mass-subcritical range $\alpha < 2$.

As a step next to small data scattering, in this article, we consider existence of a threshold solution which lies on the boundary of small scattering solutions around zero and other solutions, via concentration compactness argument. Let us make our setup more precise. We say an \hat{L}^α -solution $u(t)$ scatters forward in time (resp. backward in time) if maximal existence interval of $u(t)$ is not bounded from above (resp. from below) and if $e^{t\partial_x^3} u(t)$

converges in \hat{L}^α as $t \rightarrow \infty$ (resp. $t \rightarrow -\infty$). We define a *forward scattering set* \mathcal{S}_+ as follows

$$\mathcal{S}_+ := \left\{ u_0 \in \hat{L}^\alpha \left| \begin{array}{l} \text{a solution } u(t) \text{ to (gKdV) with } u|_{t=0} = u_0 \\ \text{scatters forward in time} \end{array} \right. \right\}.$$

A *backward scattering set* \mathcal{S}_- is defined in a similar way. We now introduce a quantity

$$(1.1) \quad \tilde{d} := \inf_{u_0 \in \hat{L}^\alpha \setminus \mathcal{S}_+} \|u_0\|_{\hat{L}^\alpha}.$$

The question we address in this article is that existence of a special solution which belongs to $\hat{L}^\alpha \setminus \mathcal{S}_+$ at each time and attains \tilde{d} in a suitable sense. By small data scattering result in [42], we know that \tilde{d} is bounded by a positive constant from below. Remark that there are several choice on notion of minimality of non-scattering solutions since $\|u(t)\|_{\hat{L}^\alpha}$ is not a conserved quantity. The above \tilde{d} is a number that gives a sharp scattering criterion; if $\|u_0\|_{\hat{L}^\alpha} < \tilde{d}$ then a corresponding solution scatters for positive time direction. However, we actually work with a weaker formulation by some technical reason (see (1.5), below).

The above problem has a connection with stability of solitons. In the focusing case (i.e., $\mu = -1$), (gKdV) admits a soliton solution

$$Q_c(t, x) = c^{\frac{1}{\alpha}} Q(c(x - c^2 t)),$$

where $Q(x)$ is a (unique) positive even solution of $-Q'' + Q = Q^{2\alpha+1}$ and $c > 0$ is a parameter describing amplitude and propagating speed of soliton. Let us remind ourselves that we consider the mass-subcritical problem. It is well known that Q is orbitally stable if $\alpha < 2$ [2, 59] and unstable if $\alpha \geq 2$ (see [3] for $\alpha > 2$ and [34] for $\alpha = 2$). When the soliton solutions are unstable, for example in the mass-critical case $\alpha = 2$, it is conjectured that the above \tilde{d} coincide with L^2 -norm (since $\alpha = 2$) of Q_c . So far, it is known that if $\alpha = 2$ then Q lies on the boundary of sets of *global* solutions and non-global solutions in H^1 , see Weinstein [58] for the sharp global existence result and Martel-Merle [35] for the existence of a finite time blow up solution.

On the other hand, in mass-subcritical case, solitons are stable (in H^1) and so they are not thresholds any longer. Indeed, it follows from [42, Theorem 1.10] that $\tilde{d} \leq c_\alpha \|Q\|_{\hat{L}^\alpha}$, where

$$(1.2) \quad c_\alpha = \left(\frac{(\alpha + 1) \|Q'\|_{L^2}^2}{\|Q\|_{L^{2\alpha+2}}^{2\alpha+2}} \right)^{\frac{1}{2\alpha}} < 1$$

is a constant such that $E[c_\alpha Q] = 0$.

Recently, there are much progress on analysis of global behavior of dispersive equations by so-called concentration compactness/rigidity argument, after a pioneering work by Kenig and Merle [25]. The existence of a critical element is one of the main step of the argument. As for generalized KdV equation (gKdV), the mass-critical case is most extensively studied in this direction. Killip-Kwon-Shao-Visan [30] constructed a minimal blow-up solution to the mass critical KdV equation in L^2 under the assumption on the space time bounds for the one dimensional mass-critical Schrödinger (NLS)

equation. Subsequently, Dodson [13] proved the global well-posedness for the one dimensional, defocusing, mass-critical NLS in L^2 . As by product of his result, the assumption imposed in [30] was removed for the defocusing case. Furthermore, Dodson [14] has shown the global well-posedness for the defocusing mass-critical KdV equation for any initial data in L^2 . For the focusing mass-critical KdV equation, Martel-Merle-Raphaël [37, 38, 39] and Martel-Merle-Nakanishi-Raphaël [36] classified the dynamics of solution into three cases (blow-up, soliton, away from soliton) in the small neighborhood of Q . As for the mass-subcritical nonlinear Schrödinger equation, the first author treated a minimization problem similar to (1.1) in a framework of weighted space and showed existence of a threshold solution which is smaller than ground state solutions (see [40, 41]).

A main contribution of this article is to extend the concentration compactness argument to \hat{L}^α -framework. We then come across two difficulties because of the fact that the \hat{L}^α -norm is invariant under the following four group actions;

- (i) Translation in physical space: $(T(a)f)(x) = f(x - a)$, $a \in \mathbb{R}$,
- (ii) Translation in Fourier space: $(P(\xi)f)(x) = e^{-ix\xi}f(x)$, $\xi \in \mathbb{R}$,
- (iii) Dilation: $(D(h)f)(x) = (D_\alpha(h)f)(x) = h^{1/\alpha}f(hx)$, $h \in 2^\mathbb{Z}$,
- (iv) Airy flow: $(A(t)f)(x) = e^{-it\partial_x^3}f(x)$, $t \in \mathbb{R}$.

They are one parameter groups of linear isometries in \hat{L}^α . In this article, we call a bijective linear isometry from a Banach space X to X itself a *deformation on X* . Further, we refer to a deformation of the form $\times\phi(x)$ as a *phase-like deformation*, and a deformation of the form $\phi((1/i)\partial_x) = \mathcal{F}^{-1}\phi(\xi)\mathcal{F}$ as a *multiplier-like deformation*, where $\phi(x) : \mathbb{R} \rightarrow \mathbb{C}$ is some function with $|\phi| = 1$. With these terminologies, $T(a) = e^{-a\partial_x}$ and $A(t)$ are multiplier-like deformations on \hat{L}^α and $P(\xi)$ is a phase-like deformation on \hat{L}^α .

The first difficulty lies in a linear profile decomposition, which is roughly speaking a decomposition of a bounded sequence of functions into a sum of characteristic profiles and a remainder by finding weak limit(s) of the sequence modulo deformations. Intuitively, this decomposition is done by a recursive use of a suitable concentration compactness result. Then, to ensure smallness of remainder as the number of detected profiles increases, a decoupling equality, so-called Pythagorean decomposition, plays a crucial role.

Let us now be more precise on the Pythagorean decomposition. Let $\{f_n\}$ be a bounded sequence of \hat{L}^α . Since \hat{L}^α is reflexive as long as $1 < \alpha < \infty$, by extracting subsequence, f_n converges to some function $f \in \hat{L}^\alpha$ in weak \hat{L}^α sense. Now we suppose that $f \neq 0$. Then, the Pythagorean decomposition is a decoupling equality of the form

$$(1.3) \quad \|\hat{f}_n\|_{L^{\alpha'}(\mathbb{R})}^{\alpha'} = \|\hat{f}\|_{L^{\alpha'}(\mathbb{R})}^{\alpha'} + \|\hat{f} - \hat{f}_n\|_{L^{\alpha'}(\mathbb{R})}^{\alpha'} + o(1)$$

as $n \rightarrow \infty$. It is well-known that the above decoupling holds for $\alpha = 2$ and may fail for $\alpha \neq 2$. Remark that the Brezis-Lieb lemma tells us that a sufficient condition for the decoupling (for $\alpha \neq 2$) is that \hat{f}_n converges to

\hat{f} almost everywhere. However, in our case, due to multiplier-like deformations T and A , which are phase-like in the Fourier side, Fourier transform of considering sequence does not necessarily converge almost everywhere. Thus, we may not expect that (1.3) holds for \hat{L}^α -norm¹. This respect is rather a serious problem for linear profile decomposition, because a decoupling like (1.3) is a key for obtaining smallness of remainder term as the number of detected profiles increases, as mentioned above.

To overcome this difficulty, we shall show a decoupling *inequality* with respect to a weaker norm, a *generalized Morrey* norm, defined as follows:

Definition 1.1. For $4/3 < \alpha < 2$ and for $\sigma \in (\alpha', \frac{6\alpha}{3\alpha-2})$, we introduce a generalized Morrey norm $\|\cdot\|_{\hat{M}_{2,\sigma}^\alpha}$ by

$$\|f\|_{\hat{M}_{2,\sigma}^\alpha} = \left\| 2^{j(\frac{1}{\alpha}-\frac{1}{2})} \|\hat{f}\|_{L^2(\tau_k^j)} \right\|_{\ell_{j,k}^\sigma} = \left\| |\tau_k^j|^{\frac{1}{2}-\frac{1}{\alpha}} \|\hat{f}\|_{L^2(\tau_k^j)} \right\|_{\ell_{j,k}^\sigma},$$

where $\tau_k^j = [k2^{-j}, (k+1)2^{-j}]$. Further, we introduce

$$(1.4) \quad \ell(u) = \ell_\sigma(u) := \inf_{\xi \in \mathbb{R}} \|P(\xi)u\|_{\hat{M}_{2,\sigma}^\alpha},$$

for $u \in \hat{L}^\alpha$.

Details on generalized Morrey space are summarized in Section 2. Here, we only note that the embedding $\hat{L}^\alpha \hookrightarrow \hat{M}_{2,\sigma}^\alpha$ holds, that $\ell(f) \sim \|f\|_{\hat{M}_{2,\sigma}^\alpha}$, and so that $\ell(f)$ is a quasi-norm and makes sense for all $f \in \hat{L}^\alpha$. It is obvious by definition that $T(a)$ and $A(t)$ are deformations on $\hat{M}_{2,\sigma}^\alpha$ for any $a, t \in \mathbb{R}$. Similarly, $D(h)$ is a deformation on $\hat{M}_{2,\sigma}^\alpha$ if h is a dyadic number. We introduce $\ell(\cdot)$ because $\hat{M}_{2,\sigma}^\alpha$ norm is not invariant (but bounded from above and below) under $P(\xi)$ action. The heart of matter is that local (in the Fourier side) L^2 norm decouples even under presence of multiplier-like deformations T and A . Hence, summing up the local L^2 decoupling with respect to intervals, we recover a decoupling *inequality* for $\ell(\cdot)$. This is one of the main ideas of this article.

Because our decoupling inequality is established only for $\ell(\cdot)$, a natural choice of the meaning of “minimality” of the solution is not with respect to $\|\cdot\|_{\hat{L}^\alpha}$ any longer but to $\ell(\cdot)$. Thus, we consider the minimization problem for

$$(1.5) \quad d_+ = d_+(\sigma, M) := \inf\{\ell(u_0) \mid u_0 \in B_M \setminus \mathcal{S}_+\},$$

where $M > 0$ is a parameter and $B_M := \{f \in \hat{L}^\alpha \mid \|f\|_{\hat{L}^\alpha} \leq M\}$ is a ball. We consider minimization problem in a ball in \hat{L}^α because well-posedness of (gKdV) is not known in the generalized Morrey space $M_{2,\sigma}^\alpha$. As a result, our threshold solution may depend on M .

Here, it is worth mentioning that the generalized Morrey space naturally appear in the context of refinement of Storchartz's estimate. The refinement, which goes back to Bourgain [4] (see also [5, 6, 45, 46, 28]), have been used for linear profile decomposition in L^2 -framework. See [43, 8, 1] for decomposition associated with Schrödinger equation and see [51] for that with Airy

¹ Actually, when $\alpha' = 4$, $f_n = f + T(n)g$ with $f, g \in \hat{L}^{4/3}$ is a counter example to the above decoupling.

equation. We show a similar refinement for a Stein-Tomas type inequality which is a version of Strichartz's estimate adopted to \hat{L}^α -framework,

$$\left\| |\partial_x|^{\frac{1}{3\alpha}} e^{-t\partial_x^3} f \right\|_{L_{t,x}^{3\alpha}(\mathbb{R} \times \mathbb{R})} \leq C \|f\|_{\dot{M}_{2,\sigma}^\alpha}.$$

For the details on this estimate, see Theorem 6.5.

The second difficulty comes from a linking between generalized KdV equation and nonlinear Schrödinger equation caused by the presence of P -deformation. More precisely, if an initial data is of the form $u_0(x) = \text{Re}[P(\xi)\phi(x)]$ then a corresponding solution to (gKdV) can be approximated in terms of a solution to nonlinear Schrödinger equation

$$(NLS) \quad \begin{cases} i\partial_t v - \partial_x^2 v = -\mu|v|^{2\alpha}v, & t, x \in \mathbb{R}, \\ v(0, x) = v_0(x), & x \in \mathbb{R}. \end{cases}$$

in the limit $|\xi| \rightarrow \infty$. This interesting phenomena is known in [30, 55] (see also [7, 10, 50]).

As for linear Airly equation, the linking with linear Schrödinger equation can be explained by an elemental identity

$$(1.6) \quad A(t)P(\xi) = e^{-it\xi^3} P(\xi)T(-3\xi^2 t)e^{3i\xi t\partial_x^2} A(t).$$

The identity infers that the presence of P on the initial data produces Schrödinger group $e^{3i\xi t\partial_x^2}$. Furthermore, in fact, the Schrödinger evolution takes a main part in the limit $|\xi| \rightarrow \infty$ because the speed of Schrödinger evolution becomes much faster than that of Airy evolution. The above identity is a kind of Galilean transform, and can be compared with the one for Schrödinger equations;

$$(1.7) \quad e^{it\partial_x^2} P(\xi) = e^{-it|\xi|^2} P(\xi)T(-2t\xi)e^{it\partial_x^2}.$$

Roughly speaking, as a nonlinear evolution generated by a class of nonlinear Schrödinger equation, such as (NLS), inherits the Galilean transform (1.7), the effect on the nonlinear problem (gKdV) which is caused by the presence of P in initial data is similar to that on the Airy equation described as in (1.6).

Because of the above linking, existence of a threshold solution is shown under the assumption

$$(1.8) \quad d_+ < 2^{1-\frac{1}{\sigma}} \left(\frac{3\sqrt{\pi}\Gamma(\alpha+2)}{2\Gamma(\alpha+\frac{3}{2})} \right)^{\frac{1}{2\alpha}} d_{NLS},$$

where d_+ is the number given in (1.5), σ is a parameter chosen to define $\ell(\cdot)$, $\Gamma(x)$ is the Gamma function, and

$$(1.9) \quad d_{NLS} = d_{NLS}(\sigma, M) := \inf\{\ell(u_0) \mid u_0 \in B_M \setminus \mathcal{S}_{NLS}\}$$

with

$$\mathcal{S}_{NLS} := \left\{ v_0 \in \hat{L}^\alpha \left| \begin{array}{l} \text{a solution } v(t) \text{ to (NLS) with } u|_{t=0} = u_0 \\ \text{scatters forward and backward in time} \end{array} \right. \right\}.$$

Here, the notion of scattering of \hat{L}^α -solution $v(t)$ to (NLS) forward in time (resp. backward in time) is defined as validity of the following two; (i) maximal existence interval of $v(t)$ is not bounded from above (resp. from below);

(ii) $e^{it\partial_x^2}v(t)$ converges in \hat{L}^α as $t \rightarrow \infty$ (resp. $t \rightarrow -\infty$). It was pointed out in [30, 55] that, in the mass-critical case $\alpha = 2$, the problem of a threshold solution for (gKdV) relates to the same problem for (NLS). Although we are working in the mass-subcritical case, the same linking appears because it is due to the presence of the P -deformation. When $\alpha = 2$, the assumption (1.8) essentially coincides with those in [30, 55].

The justification of the Schrödinger approximation is done essentially in the same way as in [30]. A key idea for dealing with nonlinearities of fractional order is to use a Fourier series expansion

$$|\cos \theta|^{2\alpha} \sin \theta = \sum_{k=1}^{\infty} C_k \sin(k\theta).$$

The constant in assumption (1.8) given in terms of the first coefficient C_1 of the expansion. For this approximation, we also establish local well-posedness of (NLS) in a scale critical \hat{L}^α space, which seems already a new result.

1.1. Main Results. In what follows, we consider the focusing case $\mu = -1$ only. However, the focusing assumption is used only for $d_+(M) < \infty$. Our analysis work also in the defocusing case $\mu = +1$ if we assume $d_+(M) < \infty$.

Theorem 1.2. *Let $3/2 + \sqrt{7/60} < \alpha < 2$ and $\sigma \in (\alpha', \frac{3\alpha(5\alpha-8)}{2(3\alpha-4)})$. Let $M > 0$ so that $B_M \cap S_+^c \neq \emptyset$. If the assumption (1.8) is true then there exists a special solution $u_c(t)$ to (gKdV) with maximal interval $I_{\max}(u_c) \ni 0$ such that*

- (i) $u_c(0) \notin S_+$;
- (ii) u_c attains d_+ in such a sense that one of the following two properties holds;
 - (a) $u_c(0) \in B_M$ and $\ell(u_c(0)) = d_+$;
 - (b) $u_c(0) \in S_-$ and scatters backward in time to $u_{c,-}$ satisfying $u_{c,-} \in B_M$ and $\ell(u_{c,-}) = d_+$.

In this article we call u_c constructed in Theorem 1.2 by *minimal non-scattering solution*.

Remark 1.3. As mentioned above, $d_+(M)$ gives a scattering criterion; if $u_0 \in \hat{L}^\alpha$ satisfies $\|u_0\|_{\hat{L}^\alpha} \leq M$ and $\ell(u_0) < d_+$ then $u_0 \in S_+$. By definition of d_+ , this is sharp in such a sense that d_+ cannot be replaced by a larger number. It is not clear whether we can replace $\ell(u_0) < d_+$ by $\ell(u_0) \leq d_+$.

The assumption $B_M \cap S_+^c \neq \emptyset$ is fulfilled for $M \geq c_\alpha \|Q\|_{\hat{L}^\alpha}$ because $c_\alpha Q \notin S_+$ by means of [42, Theorem 1.10]. By the same reason, we have the following:

Theorem 1.4. *Let $3/2 + \sqrt{7/60} < \alpha < 2$ and $\sigma \in (\alpha', \frac{3\alpha(5\alpha-8)}{2(3\alpha-4)})$. Let $M > 0$ so that $B_M \cap S_+^c \neq \emptyset$. Then, $d_+ \leq c_\alpha \ell(Q)$, where c_α is the constant given in (1.2).*

The second result is existence of minimal non-scattering solution without the assumption (1.8). For fixed $8/5 < \tilde{\alpha} < \alpha$ and $0 < \tilde{s} < 2\alpha + 1$, define

$\tilde{B}_M = \{f \in \hat{L}^\alpha \mid \|f\|_{\hat{L}^\alpha} + \|f\|_{\dot{H}^s} \leq M\}$. It turns out that, as for a minimizing problem for

$$d'_+ = d'_+(\sigma, M) := \inf\{\ell(u_0) \mid u_0 \in \tilde{B}_M \cap \mathcal{S}_+^c\},$$

a minimizer exists *without* the assumption (1.8).

Theorem 1.5. *Let $3/2 + \sqrt{7/60} < \alpha < 2$ and $\sigma \in (\alpha', \frac{3\alpha(5\alpha-8)}{2(3\alpha-4)})$. Let $M > 0$ so that $\tilde{B}_M \cap \mathcal{S}_+^c \neq \emptyset$. Then, there exists a special solution $\tilde{u}_c(t)$ to (gKdV) which attains $\tilde{d}_{+, \text{gKdV}}$ in a similar way to Theorem 1.2.*

Now let us introduce several consequential results which follow from the arguments which we establish to prove our main results. We begin with two scattering results. The first one is as follows;

Theorem 1.6. *Let $5/3 \leq \alpha < 20/9$. For any $M > 0$ there exists $\delta = \delta(M) > 0$ such that if $u_0 \in \hat{L}^\alpha$ satisfies $\|u_0\|_{\hat{L}^\alpha} \leq M$ and*

$$\left\| |\partial_x|^{\frac{1}{3\alpha}} e^{-t\partial_x^3} u_0 \right\|_{L_{t,x}^{3\alpha}(\mathbb{R} \times \mathbb{R})} \leq \delta$$

then a corresponding solution $u(t)$ to (gKdV) exists globally and scatters for both time direction.

The above theorem is a variant of small data scattering, and a consequence of a stability type estimate which is so-called long time stability. Notice that it contains the case that the data is not small in the \hat{L}^α topology.

Remark 1.7. The proof of [42, Theorem 1.7] shows that there exists a constant δ' independent of $\|u_0\|_{\hat{L}^\alpha}$ such that if

$$\left\| |\partial_x|^{\frac{1}{3\alpha}} e^{-t\partial_x^3} u_0 \right\|_{L_{t,x}^{3\alpha}(\mathbb{R} \times \mathbb{R})} + \left\| e^{-t\partial_x^3} u_0 \right\|_{L_x^{\frac{5\alpha}{2}}(\mathbb{R}, L_t^{5\alpha}(\mathbb{R}))} \leq \delta'$$

then the solution scatters for both time directions. In Theorem 1.6, smallness assumption on the second term of the left hand side is removed, however the constant δ may depends on $\|u_0\|_{\hat{L}^\alpha}$.

The second scattering result is the following.

Theorem 1.8 (Scattering due to irrelevant deformations). *Let $5/3 \leq \alpha < 2$. Let $\{u_{0,n}\}_n \subset \hat{L}^\alpha$ be a bounded sequence. Let $u_n(t)$ be a solution to (gKdV) with $u_n(0) = u_{0,n}$. If a set*

$$\left\{ \phi \in \hat{L}^\alpha \mid \begin{array}{l} \phi = \lim_{k \rightarrow \infty} (D(h_k)A(s_k)T(y_k)P(\xi_k))^{-1} u_{0,n_k} \text{ weakly in } \hat{L}^\alpha, \\ \exists (h_k, \xi_k, s_k, y_k) \in 2^{\mathbb{Z}} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \exists \text{subsequence } n_k \end{array} \right\}$$

is equal to $\{0\}$ then there exists N_0 such that $u_n(t)$ is global and scatters for both time direction as long as $n \geq N_0$.

This theorem is a consequence of Theorem 1.6 and a concentration compactness argument. An example of sequence $\{u_{0,n}\}_n$ that satisfies the assumption of Theorem 1.8 is $u_{0,n} = e^{in\partial_x^4} f$, $f \in \hat{L}^\alpha$. As a corollary, we also see that $\mathcal{S}_+ \cap \mathcal{S}_-$ is unbounded in \hat{L}^α topology.

Corollary 1.9. *For any $f \in \hat{L}^\alpha$, there exists $T > 0$ such that $e^{it\partial_x^4} f \in \mathcal{S}_+ \cap \mathcal{S}_-$ for $|t| \geq T$. In particular, $\mathcal{S}_+ \cap \mathcal{S}_-$ is an unbounded subset of \hat{L}^α .*

Unboundedness of each \mathcal{S}_+ and \mathcal{S}_- are seen by considering an orbit $\{A(t)f \mid t \in \mathbb{R}\}$ of $f \in \hat{L}^\alpha$. However, this argument does not yield that of the intersection of the both.

Finally, we state well-posedness results of nonlinear Schrödinger equation (NLS) in \hat{L}^α and $\hat{M}_{2,\sigma}^\alpha$. Although the analysis of (NLS) is not an original purpose of the article, this is necessary for our analysis because there is a linking between (gKdV) and (NLS) due to the presence of P -deformation.

Theorem 1.10 (Local well-posedness of (NLS) in \hat{L}^α). *The equation (NLS) is locally well-posed in \hat{L}^α if $4/3 < \alpha < 4$.*

Theorem 1.11 (Local well-posedness of (NLS) in $\hat{M}_{2,\sigma}^\alpha$). *The equation (NLS) is locally well-posed in $\hat{M}_{2,\sigma}^\alpha$ if $4/3 < \alpha < 2$ and $\alpha' < \sigma \leq \frac{6\alpha}{3\alpha-2}$.*

The rest of the article is organized as follows. Main theorems are proven in Section 4 after preliminaries on notations and basic facts (Section 2) and stability estimate (Section 3). For the proof, we rely on two important ingredient, linear profile decomposition (Theorem 4.3) and NLS approximation (Theorem 4.4). We prove Theorem 4.3 in Sections 5 and 6. Finally, we turn to the proof of Theorem 4.4 in Sections 7 and 8. On the other hand, consequential results are shown when we are ready; Theorem 1.6 is proven in Section 3, Theorem 1.8 is in Section 6, and Theorems 1.10 and 1.11 are in Section 7.

2. NOTATIONS AND BASIC FACTS

In this section, we introduce several notations and give lemmas which are needed to prove main results.

The following notation will be used throughout this paper: $|\partial_x|^s = (-\partial_x^2)^{s/2}$ denotes the Riesz potential of order $-s$. For $1 \leq p, q \leq \infty$ and $I \subset \mathbb{R}$, let us define a space-time norm

$$\|f\|_{L_x^p L_t^q(I)} = \| \|f(\cdot, x)\|_{L_t^q(I)} \|_{L_x^p(\mathbb{R})}.$$

2.1. Deformations. Let us first collect elementary facts on the deformations which is used thorough out the article. As in the introduction, we set

- $(T(y)f)(x) = f(x - y), \quad y \in \mathbb{R},$
- $(P(\xi)f)(x) = e^{-ix\xi} f(x), \quad \xi \in \mathbb{R},$
- $(D_p(h)f)(x) = h^{1/p} f(hx), \quad h \in 2^\mathbb{Z},$
- $(A(t)f)(x) = e^{-it\partial_x^3} f(x), \quad t \in \mathbb{R}.$

They are deformations on \hat{L}^p for any $1 \leq p \leq \infty$. Denote $D(h) = D_\alpha(h)$, where α is the number in (gKdV). Let $S(t) = e^{it\partial_x^2}$ be a Schrödinger group. Notice that $S(t)$ is also a deformation on \hat{L}^p , $1 \leq p \leq \infty$. The inverses of $A(t)$, $S(t)$, $T(y)$, and $P(\xi)$ are $A(-t)$, $S(-t)$, $T(-y)$, and $P(-\xi)$, respectively. Further, $D_p(h)^{-1} = D_p(h^{-1})$.

We use a notation $\hat{X} := \mathcal{F}X\mathcal{F}^{-1}$, or equivalently, $\mathcal{F}X = \hat{X}\mathcal{F}$, for $X = A, S, T, P, D$. More specifically, $\hat{A}(t) = e^{it\xi^3}$, $\hat{S}(t) = e^{-it\xi^2}$, $\hat{T}(y) = P(y)$, $\hat{P}(\xi) = T(-\xi)$, and $\hat{D}_\alpha(h) = D_{\alpha'}(h^{-1})$. With this notation, the identity

(1.6) is easily obtained as follows.

$$\hat{P}(\xi_0)^{-1} \hat{A}(t) \hat{P}(\xi_0) = e^{it(\xi-\xi_0)^3} = e^{-i\xi_0^3 t} \hat{T}(-3\xi_0^2 t) \hat{S}(3\xi_0 t) \hat{A}(t).$$

Next, we collect commutations of the above deformations. We have

$$[A(t), S(t)] = [A(t), T(y)] = [S(t), T(y)] = 0, \quad T(y)P(\xi) = e^{iy\xi} P(\xi)T(y).$$

Commutation property for $D(h)$ is as follows:

$$\begin{aligned} A(t)D(h) &= D(h)A(h^3 t), & S(t)D(h) &= D(h)S(h^2 t), \\ T(y)D(h) &= D(h)T(hy), & P(\xi)D(h) &= D(h)P(h^{-1}\xi). \end{aligned}$$

Combining above relations, we have the following identity

$$\begin{aligned} (2.1) \quad & (D(\tilde{h})T(\tilde{y})A(\tilde{s})P(\tilde{\xi}))^{-1} (D(h)T(y)A(s)P(\xi)) \\ &= e^{i\gamma} D\left(\frac{h}{\tilde{h}}\right) P\left(\xi - \frac{\tilde{h}}{h}\tilde{\xi}\right) A\left(s - \left(\frac{h}{\tilde{h}}\right)^3 \tilde{s}\right) \\ & S\left(3\left(s - \left(\frac{h}{\tilde{h}}\right)^3 \tilde{s}\right)\xi\right) T\left(y - \frac{h}{\tilde{h}}\tilde{y} - 3\left(s - \left(\frac{h}{\tilde{h}}\right)^3 \tilde{s}\right)\xi^2\right), \end{aligned}$$

where γ is a real number given by $h, y, s, \xi, \tilde{h}, \tilde{y}, \tilde{s}, \tilde{\xi}$. This identity is useful for linear profile decomposition (see Remark 4.2).

2.2. Generalized Morrey space. For $j \in \mathbb{Z}$, we set $\mathcal{D}_j := \{[k2^{-j}, (k+1)2^{-j}) \mid k \in \mathbb{Z}\}$. Let $\mathcal{D} := \cup_{j \in \mathbb{Z}} \mathcal{D}_j$. For a function $a : \mathcal{D} \rightarrow \mathbb{C}$, we denote $\|a\|_{\ell_r^{\mathcal{D}}} := (\sum_{I \in \mathcal{D}} |a(I)|^r)^{1/r}$ if $0 < r < \infty$ and $\|a\|_{\ell_\infty^{\mathcal{D}}} := \sup_{I \in \mathcal{D}} |a(I)|$.

Definition 2.1. For $1 \leq q \leq p \leq \infty$ and for $r \in [1, \infty]$, we introduce a generalized Morrey norm $\|\cdot\|_{M_{q,r}^p}$ by

$$\|f\|_{M_{q,r}^p} = \left\| |I|^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^q(I)} \right\|_{\ell_r^{\mathcal{D}}}.$$

Here, the case $p = q$ and $r < \infty$ is excluded. For $1 \leq p \leq q \leq \infty$ and for $r \in [1, \infty]$, we also introduce $\|f\|_{\hat{M}_{q,r}^p} := \|\hat{f}\|_{M_{q',r}^{p'}}$, i.e.,

$$\|f\|_{\hat{M}_{q,r}^p} = \left\| |I|^{\frac{1}{q}-\frac{1}{p}} \|\hat{f}\|_{L^{q'}(I)} \right\|_{\ell_r^{\mathcal{D}}}.$$

Banach spaces $M_{q,r}^p$ and $\hat{M}_{q,r}^p$ are defined as sets of tempered distributions of which above norms are finite, respectively.

- Remark 2.2.* (i) $M_{q,\infty}^p$ is a usual Morrey space. $M_{p,\infty}^p = L^p$ with equal norm.
(ii) For any $1 \leq q_1 \leq q_2 \leq p \leq \infty$ and $1 \leq r_1 \leq r_2 \leq \infty$, it holds that $M_{q_1,r_1}^p \hookrightarrow M_{q_2,r_2}^p$.
(iii) For any $1 \leq p \leq q_2 \leq q_1 \leq \infty$ and $1 \leq r_1 \leq r_2 \leq \infty$, it holds that $\hat{M}_{q_1,r_1}^p \hookrightarrow \hat{M}_{q_2,r_2}^p$.
(iv) $L^p \hookrightarrow M_{q,r}^p$ holds as long as $1 \leq q < p < r \leq \infty$.
(v) $\hat{L}^p \hookrightarrow \hat{M}_{q,r}^p$ holds as long as $1 \leq q' < p' < r \leq \infty$.

For the last two assertions, see Proposition A.1.

Lemma 2.3. *Let $1 \leq p \leq q \leq \infty$ and let $r \in (0, \infty]$. There exists a constant $C \geq 1$ such that*

$$C^{-1} \|f\|_{\hat{M}_{q,r}^p} \leq \|D_p(h)A(s)T(y)P(\xi)f\|_{\hat{M}_{q,r}^p} \leq C \|f\|_{\hat{M}_{q,r}^p}$$

for any $f \in \hat{M}_{q,r}^p$ and any $(h, \xi, s, y) \in 2^{\mathbb{Z}} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. Further, if $\xi = 0$ then the above inequality hold with $C = 1$.

Proof. We only consider $q > 1$. Notice that

$$|\mathcal{F}D_p(h)A(s)T(y)P(\xi)f|(x) = h^{-\frac{1}{p'}} |\mathcal{F}f|\left(\frac{x}{h} + \xi\right).$$

Therefore, for any $\tau_k^j = [k/2^j, (k+1)/2^j) \in \mathcal{D}_j$ we have

$$|\tau_k^j|^{\frac{1}{p'} - \frac{1}{q'}} \|\mathcal{F}D_p(h)A(s)T(y)P(\xi)f\|_{L^{q'}(\tau_k^j)} = |\tilde{\tau}_k^j|^{\frac{1}{p'} - \frac{1}{q'}} \|\mathcal{F}f\|_{L^{q'}(\tilde{\tau}_k^j)},$$

where

$$\tilde{\tau}_k^j = \left[\frac{k}{h2^j} + \xi, \frac{k+1}{h2^j} + \xi \right).$$

Denote $h = 2^{j_0}$. We choose $k_0 = k_0(j)$ so that $k_0 \leq 2^{j+j_0}\xi < k_0 + 1$. Then,

$$\tilde{\tau}_k^j = \left[\frac{k + 2^{j+j_0}\xi}{2^{j+j_0}}, \frac{k + 2^{j+j_0}\xi + 1}{2^{j+j_0}} \right) \subset \tau_{k+k_0}^{j+j_0} \cup \tau_{k+k_0+1}^{j+j_0}$$

and $|\tilde{\tau}_k^j| = |\tau_{k+k_0}^{j+j_0}| = |\tau_{k+k_0+1}^{j+j_0}|$. Thus,

$$\begin{aligned} & |\tilde{\tau}_k^j|^{\frac{1}{p'} - \frac{1}{q'}} \|\mathcal{F}f\|_{L^{q'}(\tilde{\tau}_k^j)} \\ & \leq |\tilde{\tau}_k^j|^{\frac{1}{p'} - \frac{1}{q'}} \left(\|\mathcal{F}f\|_{L^{q'}(\tau_{k+k_0}^{j+j_0})}^{q'} + \|\mathcal{F}f\|_{L^{q'}(\tau_{k+k_0+1}^{j+j_0})}^{q'} \right)^{\frac{1}{q'}} \\ & \leq |\tau_{k+k_0}^{j+j_0}|^{\frac{1}{p'} - \frac{1}{q'}} \|\mathcal{F}f\|_{L^{q'}(\tau_{k+k_0}^{j+j_0})} + |\tau_{k+k_0+1}^{j+j_0}|^{\frac{1}{p'} - \frac{1}{q'}} \|\mathcal{F}f\|_{L^{q'}(\tau_{k+k_0+1}^{j+j_0})}. \end{aligned}$$

We take ℓ_k^r norm and then ℓ_j^r norm to obtain the second inequality with $C = 2$. It is obvious that if $\xi = 0$ then $\tilde{\tau}_k^j = \tau_k^{j+j_0}$ and $|\tau_k^j|/h = |\tau_k^{j+j_0}|$ hold and so we can take $C = 1$. The first inequality follows in the same way. We repeat the same argument from $|\mathcal{F}f|(y) = |\mathcal{F}D(h)A(s)T(y)P(\xi)f|(hy - h\xi)$. \square

2.3. Generalized Strichartz's estimates. In this subsection we give a generalized Strichartz's estimates for the Airy equation. To this end, we introduce several notations.

Definition 2.4. (i) A pair $(s, r) \in \mathbb{R} \times [1, \infty]$ is said to be acceptable if $1/r \in [0, 3/4]$ and

$$s \in \begin{cases} [-\frac{1}{2r}, \frac{2}{r}] & 0 \leq \frac{1}{r} \leq \frac{1}{2}, \\ (\frac{2}{r} - \frac{5}{4}, \frac{5}{2} - \frac{3}{r}) & \frac{1}{2} < \frac{1}{r} < \frac{3}{4}. \end{cases}$$

(ii) A pair $(s, r) \in \mathbb{R} \times [1, \infty]$ is said to be conjugate-acceptable if $(1-s, r')$ is acceptable, where $\frac{1}{r'} = 1 - \frac{1}{r} \in [0, 1]$.

For an interval $I \subset \mathbb{R}$ and an acceptable pair (s, r) , we define a function space $X(I; s, r)$ of space-time functions with the following norm

$$\|f\|_{X(I; s, r)} = \| |\partial_x|^s f \|_{L_x^{p(s, r)}(\mathbb{R}; L_t^{q(s, r)}(I))},$$

where the exponents $p(s, r)$ and $q(s, r)$ are given by

$$(2.2) \quad \frac{2}{p(s, r)} + \frac{1}{q(s, r)} = \frac{1}{r}, \quad -\frac{1}{p(s, r)} + \frac{2}{q(s, r)} = s,$$

or equivalently,

$$\begin{pmatrix} 1/p(s, r) \\ 1/q(s, r) \end{pmatrix} = \begin{pmatrix} -1/5 & 2/5 \\ 2/5 & 1/5 \end{pmatrix} \begin{pmatrix} s \\ 1/r \end{pmatrix}.$$

We refer $X(I; s, r)$ to as an \hat{L}^r -admissible space.

For an interval $I \subset \mathbb{R}$ and a conjugate-acceptable pair (s, r) , we define a function space $Y(I; s, r)$ by

$$\|f\|_{Y(I; s, r)} = \| |\partial_x|^s f \|_{L_x^{\tilde{p}(s, r)}(\mathbb{R}; L_t^{\tilde{q}(s, r)}(I))},$$

where the exponents $\tilde{p}(s, r)$ and $\tilde{q}(s, r)$ are given by

$$(2.3) \quad \frac{2}{\tilde{p}(s, r)} + \frac{1}{\tilde{q}(s, r)} = 2 + \frac{1}{r}, \quad -\frac{1}{\tilde{p}(s, r)} + \frac{2}{\tilde{q}(s, r)} = s,$$

or equivalently,

$$\begin{pmatrix} 1/\tilde{p}(s, r) \\ 1/\tilde{q}(s, r) \end{pmatrix} = \begin{pmatrix} -1/5 & 2/5 \\ 2/5 & 1/5 \end{pmatrix} \begin{pmatrix} s \\ 2 + 1/r \end{pmatrix} = \begin{pmatrix} 1/p(s, r) \\ 1/q(s, r) \end{pmatrix} + \begin{pmatrix} 4/5 \\ 2/5 \end{pmatrix}.$$

Let us define some specific $X(I; s, r)$ and $Y(I; s, r)$ type spaces by choosing specific degrees $s = s(r)$.

Definition 2.5. Set $s(L) = s(L, \alpha) := 1/(3\alpha)$, $s(Z) = s(Z, \alpha) := \frac{5}{2} - \frac{3}{\alpha} - \varepsilon$, and $s(K) = s(K, r) := \frac{2}{\alpha} - \frac{5}{4} + \varepsilon$, where $\varepsilon > 0$ is a sufficiently small number. Define $S(I) := X(I; 0, \alpha)$, $L(I) := X(I; s(L), \alpha)$, $Z(I) := X(I; s(Z), \alpha)$, and $K(I) := X(I; s(K), \alpha)$. Also define $N(I) := Y(I; s(L), \alpha)$. We use the notation $(p(X), q(X)) := (p(s(X), \alpha), q(s(X), \alpha))$ for $X = S, L, K, Z$ and $(\tilde{p}(N), \tilde{q}(N)) = (\tilde{p}(s(L), \alpha), \tilde{q}(s(L), \alpha))$.

From the definition, we have $(p(S), q(S)) = (\frac{5}{2}\alpha, 5\alpha)$ and $(p(L), q(L)) = (3\alpha, 3\alpha)$. For details of choice of $s(Z)$ and $s(K)$, see Remark 4.12 below.

Remark 2.6. The $S(I)$ norm is so-called *scattering norm*. This norm plays an important role on well-posedness theory. For example, criterion for blowup and scattering are given in terms of the scattering norm (See [42, Theorems 1.8 and 1.9]). Notice that the pair $(0, \alpha)$ is admissible only if $\alpha > 8/5$. The $L(I)$ norm is a non-mixed space. This norm appears in refinement of Stein-Tomas type inequality, see Theorem 6.5, below. A pair $(s_L(\alpha), \alpha)$ is acceptable and conjugate-acceptable if $5/3 \leq \alpha < 20/9$. Remark that there exists an acceptable and conjugate-acceptable pair under a weaker assumption $10/7 < \alpha < 10/3$ (see [42, Remark 4.1]).

We have the following generalized version of Strichartz's estimate.

Proposition 2.7 (Generalized Strichartz's estimates).

(i) (homogeneous estimate) It holds for any acceptable pair (s, r) and interval I that

$$(2.4) \quad \left\| e^{-t\partial_x^3} f \right\|_{X(I; s, r)} \leq C \|f\|_{\dot{L}^r},$$

where the constant C depends only on s and r .

(ii) (inhomogeneous estimate) Let $4/3 < r < 4$. Let (s_1, r) be an acceptable pair and (s_2, r) be a conjugate-acceptable pair. Then, it holds for any $t_0 \in I \subset \mathbb{R}$ that

$$(2.5) \quad \left\| \int_{t_0}^t e^{-(t-t')\partial_x^3} \partial_x F(t') dt' \right\|_{L_t^\infty(I; \dot{L}_x^r) \cap X(I, s_1, r)} \leq C \|F\|_{Y(I, s_2, r)},$$

where the constant C depends on s_1, s_2 and r .

Proof. The inequality (2.4) is obtained by interpolating the notable Kato's smoothing effect, the Kenig-Ruiz estimate and the Stein-Tomas inequality. See [42, Proposition 2.1] for the detail. Moreover, the inhomogeneous estimate (2.5) follows from the combination of the homogeneous inequality (2.4) and the Christ-Kiselev lemma. See [42, Proposition 2.5] for the detail. \square

To handle $X(I; s, r)$ and $Y(I; s, r)$ spaces, the following lemma is useful.

Lemma 2.8. Let $1 < p_i, q_i < \infty$ and $s_i \in \mathbb{R}$ for $i = 1, 2$. Let p, q, s be

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \quad \frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}, \quad s = \theta s_1 + (1-\theta)s_2$$

for some $\theta \in (0, 1)$. Then, there exists a positive constant C such that the inequality

$$\| |\partial_x|^s f \|_{L_x^p L_t^q} \leq C \| |\partial_x|^{s_1} f \|_{L_x^{p_1} L_t^{q_1}}^\theta \| |\partial_x|^{s_2} f \|_{L_x^{p_2} L_t^{q_2}}^{1-\theta}$$

holds for any f such that $|\partial_x|^{s_1} f \in L_x^{p_1} L_t^{q_1}$ and $|\partial_x|^{s_2} f \in L_x^{p_2} L_t^{q_2}$.

Proof. See [42, Lemma 3.3]. \square

To evaluate the nonlinear term, we need the following lemma.

Lemma 2.9. Suppose that $8/5 < \alpha < 10/3$. Let (s, r) be a pair which is acceptable and conjugate-acceptable. Then, the following two assertions hold:

(i) If $u \in S(I) \cap X(I; s, r)$ then $|u|^{2\alpha} u \in Y(I; s, r)$. Moreover, there exists a positive constant C such that the inequality

$$\| |u|^{2\alpha} u \|_{Y(I; s, r)} \leq C \|u\|_{S(I)}^{2\alpha} \|u\|_{X(I; s, r)}$$

holds for any $u \in S(I) \cap X(I; s, r)$.

(ii) There exists a positive constant C such that the inequality

$$\begin{aligned} & \| |u|^{2\alpha} u - |v|^{2\alpha} v \|_{Y(I; s, r)} \\ & \leq C (\|u\|_{X(I; s, r)} + \|v\|_{X(I; s, r)}) (\|u\|_{S(I)} + \|v\|_{S(I)})^{2\alpha-1} \|u - v\|_{S(I)} \\ & \quad + C (\|u\|_{S(I)} + \|v\|_{S(I)})^{2\alpha} \|u - v\|_{X(I; s, r)} \end{aligned}$$

holds for any $u, v \in S(I) \cap X(I; s, r)$.

Proof. See [42, Proposition 3.4]. \square

3. STABILITY ESTIMATES

3.1. Stability for gKdV. We consider the generalized KdV equation with the perturbation:

$$(3.1) \quad \begin{cases} \partial_t \tilde{u} + \partial_x^3 \tilde{u} = \mu \partial_x (|\tilde{u}|^{2\alpha} \tilde{u}) + \partial_x e, & t, x \in \mathbb{R}, \\ \tilde{u}(\hat{t}, x) = \tilde{u}_0(x), & x \in \mathbb{R}, \end{cases}$$

where the perturbation e is small in a suitable sense and the initial data \tilde{u}_0 is close to u_0 .

The estimates in this section are restricted to $5/3 \leq \alpha < 20/9$ but one can easily extend the results for $8/5 < \alpha < 10/3$ by modifying the definitions of $L(I)$ and $N(I)$ spaces. See Remark 2.6 for the meaning of the above restriction on α .

Lemma 3.1 (Short time stability for gKdV). *Assume $5/3 \leq \alpha < 20/9$ and $\hat{t} \in \mathbb{R}$. Let I be a time interval containing \hat{t} and let \tilde{u} be a solution to (3.1) on $I \times \mathbb{R}$ for some function e . Then, there exists $\varepsilon_0 > 0$ such that if \tilde{u} and e satisfy*

$$\|\tilde{u}\|_{S(I)} + \|\tilde{u}\|_{L(I)} \leq \varepsilon_0,$$

and

$$\|e^{-(t-\hat{t})\partial_x^3}(u(\hat{t}) - \tilde{u}(\hat{t}))\|_{S(I)} + \|e^{-(t-\hat{t})\partial_x^3}(u(\hat{t}) - \tilde{u}(\hat{t}))\|_{L(I)} + \|e\|_{N(I)} \leq \varepsilon,$$

and if $0 < \varepsilon < \varepsilon_0$ hold, then there exists a unique solution $u \in S(I) \cap L(I)$ to (gKdV) satisfying

$$(3.2) \quad \|u - \tilde{u}\|_{S(I)} + \|u - \tilde{u}\|_{L(I)} \leq C\varepsilon,$$

$$(3.3) \quad \| |u|^{2\alpha} u - |\tilde{u}|^{2\alpha} \tilde{u} \|_{N(I)} \leq C\varepsilon,$$

If further $u(\hat{t}) - \tilde{u}(\hat{t}) \in \hat{L}^\alpha$ holds then

$$(3.4) \quad \|u - \tilde{u}\|_{L_t^\infty(I; \hat{L}_x^\alpha)} \leq \|u(\hat{t}) - \tilde{u}(\hat{t})\|_{\hat{L}_x^\alpha} + C\varepsilon.$$

Proof. By the local well-posedness theory, it suffices to show (3.2), (3.3), and (3.4) as a priori estimates. Let $w := u - \tilde{u}$. Then w satisfies

$$\begin{aligned} w(t) &= e^{-(t-\hat{t})\partial_x^3} w(\hat{t}) + \mu \int_{\hat{t}}^t e^{-(t-t')\partial_x^3} \partial_x \{ |\tilde{u} + w|^{2\alpha} (\tilde{u} + w) - |\tilde{u}|^{2\alpha} \tilde{u} \} dt' \\ &\quad - \int_{\hat{t}}^t e^{-(t-t')\partial_x^3} \partial_x e(t') dt'. \end{aligned}$$

For $t \in I$, set

$$F(t) := \|w\|_{S(0,t)} + \|w\|_{L(0,t)},$$

$$G(t) := \| |\tilde{u} + w|^{2\alpha} (\tilde{u} + w) - |\tilde{u}|^{2\alpha} \tilde{u} \|_{N(0,t)},$$

where we use abbreviation such as $S(0, t) = S([0, t])$ to simplify notation. Then the assumptions on $u(\hat{t})$, $\tilde{u}(\hat{t})$ and e , and Proposition 2.7 (2.5) lead us to

$$\begin{aligned} F(t) &\leq \|e^{-(t-\hat{t})\partial_x^3}(u(\hat{t}) - \tilde{u}(\hat{t}))\|_{S(0,t)} + \|e^{-(t-\hat{t})\partial_x^3}(u(\hat{t}) - \tilde{u}(\hat{t}))\|_{L(0,t)} \\ &\quad + CG(t) + C\|e\|_{N(0,t)} \\ &\leq C\varepsilon + CG(t). \end{aligned}$$

Lemma 2.9 (ii) yields

$$\begin{aligned}
 (3.5) \quad G(t) &\leq C(\|\tilde{u} + w\|_{L(0,t)} + \|\tilde{u}\|_{L(0,t)}) \\
 &\quad \times (\|\tilde{u} + w\|_{S(0,t)} + \|\tilde{u}\|_{S(0,t)})^{2\alpha-1} \|w\|_{S(0,t)} \\
 &\quad + C(\|\tilde{u} + w\|_{S(0,t)} + \|\tilde{u}\|_{S(0,t)})^{2\alpha} \|w\|_{L(0,t)} \\
 &\leq C(\varepsilon_0 + F(t))^{2\alpha} F(t).
 \end{aligned}$$

Hence

$$F(t) \leq C\varepsilon + C\varepsilon_0^{2\alpha} F(t) + CF(t)^{2\alpha+1}.$$

Since $F(0) = 0$, by the continuity argument, we have that if $C\varepsilon_0^{2\alpha} < 1$, then $F(t) \leq C\varepsilon$ for any $t \in I$. Hence we have (3.2). Combining (3.2) and (3.5), we have (3.3).

Now we suppose that $w(\hat{t}) = u(\hat{t}) - \tilde{u}(\hat{t}) \in \hat{L}^\alpha$. Then, Proposition 2.7 (2.4) and (2.5) yield

$$\begin{aligned}
 &\|w\|_{L_t^\infty(I)\hat{L}_x^\alpha} \\
 &\leq \|w(\hat{t})\|_{\hat{L}_x^\alpha} + C\|\tilde{u} + w\|^{2\alpha}(\tilde{u} + w) - |\tilde{u}|^{2\alpha}\tilde{u}\|_{N(I)} + C\|e\|_{N(I)} \\
 &\leq \|w(\hat{t})\|_{\hat{L}_x^\alpha} + C\varepsilon,
 \end{aligned}$$

which is (3.4). This completes the proof of Lemma 3.1. \square

Proposition 3.2 (Long time stability for gKdV). *Assume $5/3 \leq \alpha < 20/9$ and $\hat{t} \in \mathbb{R}$. Let $I \subset \mathbb{R}$ be an interval containing \hat{t} . Let \tilde{u} be a solution to (3.1) on $I \times \mathbb{R}$ for some function e . Assume that \tilde{u} satisfies*

$$\|\tilde{u}\|_{S(I)} + \|\tilde{u}\|_{L(I)} \leq M,$$

for some $M > 0$. Then there exists $\varepsilon_1 = \varepsilon_1(M) > 0$ such that if

$$\|e^{-t\partial_x^3}(u(\hat{t}) - \tilde{u}(\hat{t}))\|_{S(I)} + \|e^{-t\partial_x^3}(u(\hat{t}) - \tilde{u}(\hat{t}))\|_{L(I)} + \|e\|_{N(I)} \leq \varepsilon$$

and $0 < \varepsilon < \varepsilon_1$, then there exists a solution u to (gKdV) on $I \times \mathbb{R}$ satisfies

$$(3.6) \quad \|u - \tilde{u}\|_{S(I)} + \|u - \tilde{u}\|_{L(I)} \leq C\varepsilon,$$

$$(3.7) \quad \| |u|^{2\alpha}u - |\tilde{u}|^{2\alpha}\tilde{u} \|_{N(I)} \leq C\varepsilon,$$

where the constant C depends only on M . Further, if $u(\hat{t}) - \tilde{u}(\hat{t}) \in \hat{L}^\alpha$ for some $\hat{t} \in I$ then, it also holds that

$$(3.8) \quad \|u - \tilde{u}\|_{L_t^\infty(I;\hat{L}_x^\alpha)} \leq \|u(\hat{t}) - \tilde{u}(\hat{t})\|_{\hat{L}^\alpha} + C\varepsilon.$$

Proof. The proof is the combination of Lemma 3.1 and an iterative procedure. Without loss of generality, we may assume that $\hat{t} = 0$ and $\inf I = 0$. Now let ε_0 be the constant given in Lemma 3.1. We first show the following claim: There exists a positive integer $N \leq 1 + (2M/\varepsilon_0)^{q(S)}$ such that

$$I = \bigcup_{j=1}^N I_j, \quad I_j = [t_{j-1}, t_j] \quad \text{with} \quad \|\tilde{u}\|_{S(I_j)} + \|\tilde{u}\|_{L(I_j)} \leq \varepsilon_0$$

for any $1 \leq j \leq N$. Suppose $M > \varepsilon_0$, otherwise there is nothing to prove. Take $t_1 \in I$ so that $t_0 < t_1$ and $\|\tilde{u}\|_{S(I_1)} + \|\tilde{u}\|_{L(I_1)} = \varepsilon_0$. Similarly, as long as $\|\tilde{u}\|_{S((t_{j-1}, \sup I))} + \|\tilde{u}\|_{L((t_{j-1}, \sup I))} > \varepsilon_0$ we define $t_j \in I$ so that $t_{j-1} < t_j$ and $\|\tilde{u}\|_{S(I_j)} + \|\tilde{u}\|_{L(I_j)} = \varepsilon_0$. Now we show that $N \leq 1 + (2M/\varepsilon_0)^{q(S)}$ by the

contradiction argument. Suppose that $1 + (2M/\varepsilon_0)^{q(S)} < N \leq \infty$. Let N' be an integer defined by $N' = N$ if N is finite and N' any integer satisfying $1 + (2M/\varepsilon_0)^{q(S)} < N'$ if N is infinite.

For $1 \leq j \leq N'$, set

$$f_j(x) := \|\tilde{u}(\cdot, x)\|_{L_t^{q(S)}(I_j)}, \quad g_j(x) := \|\partial_x |s(L)| \tilde{u}(\cdot, x)\|_{L_t^{q(L)}(I_j)}.$$

Then

$$\begin{aligned} (3.9) \quad M &\geq \|\tilde{u}\|_{S((0, t_{N'}))} = \left\| \left(\|\tilde{u}(\cdot, x)\|_{L_t^{q(S)}((0, t_{N'}))}^{q(S)} \right)^{\frac{1}{q(S)}} \right\|_{L_x^{p(S)}} \\ &= \left\| \left(\sum_{j=1}^{N'} |f_j(x)|^{q(S)} \right)^{\frac{1}{q(S)}} \right\|_{L_x^{p(S)}}. \end{aligned}$$

In a similar way, we have

$$(3.10) \quad M \geq \left\| \left(\sum_{j=1}^{N'} |g_j(x)|^{q(L)} \right)^{\frac{1}{q(L)}} \right\|_{L_x^{p(L)}}.$$

Noting $p(S) < q(S)$ and $p(L) = q(L)$, by the Hölder inequality, (3.9) and (3.10), we obtain

$$\begin{aligned} \varepsilon_0 N' &= \sum_{j=1}^{N'} (\|\tilde{u}\|_{S(I_j)} + \|\tilde{u}\|_{L(I_j)}) \\ &\leq (N')^{1-\frac{1}{p(S)}} \left(\sum_{j=1}^{N'} \|\tilde{u}\|_{S(I_j)}^{p(S)} \right)^{\frac{1}{p(S)}} + (N')^{1-\frac{1}{p(L)}} \left(\sum_{j=1}^{N'} \|\tilde{u}\|_{L(I_j)}^{p(L)} \right)^{\frac{1}{p(L)}} \\ &= (N')^{1-\frac{1}{p(S)}} \left\| \left(\sum_{j=1}^{N'} |f_j(x)|^{p(S)} \right)^{\frac{1}{p(S)}} \right\|_{L_x^{p(S)}} \\ &\quad + (N')^{1-\frac{1}{p(L)}} \left\| \left(\sum_{j=1}^{N'} |g_j(x)|^{p(L)} \right)^{\frac{1}{p(L)}} \right\|_{L_x^{p(L)}} \\ &\leq (N')^{1-\frac{1}{q(S)}} \left\| \left(\sum_{j=1}^{N'} |f_j(x)|^{q(S)} \right)^{\frac{1}{q(S)}} \right\|_{L_x^{p(S)}} \\ &\quad + (N')^{1-\frac{1}{q(L)}} \left\| \left(\sum_{j=1}^{N'} |g_j(x)|^{q(L)} \right)^{\frac{1}{q(L)}} \right\|_{L_x^{p(L)}} \\ &\leq ((N')^{1-\frac{1}{q(S)}} + (N')^{1-\frac{1}{q(L)}}) M. \end{aligned}$$

Since $q(L) > q(S)$, we obtain $N' \leq 1 + (2M/\varepsilon_0)^{q(L)}$. This contradicts the definition of N' , which proves the claim.

From Lemma 3.1, we have that there exists a positive constant C_0 such that if a positive constant η_j satisfies

$$(3.11) \quad \|e^{-(t-t_{j-1})\partial_x^3} w(t_{j-1})\|_{S(I_j)} + \|e^{-(t-t_{j-1})\partial_x^3} w(t_{j-1})\|_{L(I_j)} \leq \eta_j,$$

and

$$(3.12) \quad \eta_j \leq \varepsilon_0,$$

then we have

$$(3.13) \quad \|w\|_{S(I_j)} + \|w\|_{L(I_j)} \leq C_0 \eta_j,$$

$$(3.14) \quad \| |u|^{2\alpha} u - |\tilde{u}|^{2\alpha} \tilde{u} \|_{N(I_j)} \leq C_0 \eta_j.$$

On the other hand, since $w(t_{j-1})$ satisfies the integral equation

$$\begin{aligned} & e^{-(t-t_{j-1})\partial_x^3} w(t_{j-1}) \\ &= e^{-t\partial_x^3} w(0) + \mu \int_0^{t_{j-1}} e^{-(t-t')\partial_x^3} \partial_x (|u|^{2\alpha} u - |\tilde{u}|^{2\alpha} \tilde{u}) dt' \\ & \quad - \int_0^{t_{j-1}} e^{-(t-t')\partial_x^3} \partial_x e(t') dt', \end{aligned}$$

Proposition 2.5 and the assumption yield

$$\begin{aligned} (3.15) \quad & \|e^{-(t-t_{j-1})\partial_x^3} w(t_{j-1})\|_{S(I_j)} + \|e^{-(t-t_{j-1})\partial_x^3} w(t_{j-1})\|_{L(I_j)} \\ & \leq \|e^{-t\partial_x^3} w(0)\|_{S(I_j)} + \|e^{-t\partial_x^3} w(0)\|_{L(I_j)} \\ & \quad + C \| |u|^{2\alpha} u - |\tilde{u}|^{2\alpha} \tilde{u} \|_{N(0,t_{j-1})} + \|e\|_{N(0,t_{j-1})} \\ & \leq C_1 \varepsilon + C_1 \sum_{k=1}^{j-1} \| |u|^{2\alpha} u - |\tilde{u}|^{2\alpha} \tilde{u} \|_{N(I_j)}. \end{aligned}$$

Let $\beta \geq C_0 C_1 + 1$ and let $\eta_j := \beta^{j-1} C_1 \varepsilon$ for $1 \leq j \leq N$. Then we easily see that $\eta_1 < \eta_2 < \dots < \eta_N = \beta^{N-1} C_1 \varepsilon$. Here, we take

$$\varepsilon_1 := C_1^{-1} \beta^{-N+1} \varepsilon_0.$$

Then, we easily see that η_j satisfies (3.12) for all $1 \leq j \leq N$ if $\varepsilon \leq \varepsilon_1$. Let us show that if $\varepsilon \leq \varepsilon_1$, then (3.11) also holds for $1 \leq j \leq N$, by an induction argument on j , which yields (3.13) and (3.14) hold for $1 \leq j \leq N$.

For $j = 1$, (3.11) is fulfilled by the assumption. Assume for induction that (3.11) hold for $1 \leq j \leq J$ ($1 \leq J \leq N-1$). Since (3.14) holds for $1 \leq j \leq J$, from (3.15), we have

$$\begin{aligned} & \|e^{-(t-t_J)\partial_x^3} (u(t_J) - \tilde{u}(t_J))\|_{S(I_{J+1})} \\ & \quad + \|e^{-(t-t_J)\partial_x^3} (u(t_J) - \tilde{u}(t_J))\|_{L(I_{J+1})} \\ & \leq C_1 \varepsilon + C_0 C_1 \sum_{k=1}^J \eta_k \leq \eta_1 + (\beta - 1) \sum_{k=1}^J \eta_k \\ & = \beta^{-J} \eta_{J+1} + (\beta - 1) \sum_{k=1}^J \beta^{k-J-1} \eta_{J+1} = \eta_{J+1}. \end{aligned}$$

Therefore (3.11) holds for $j = J + 1$, and hence for all $1 \leq j \leq N$ by induction.

From (3.13), we have

$$\|w\|_{S(I)} + \|w\|_{L(I)} \leq \sum_{j=1}^N (\|w\|_{S(I_j)} + \|w\|_{L(I_j)}) \leq C_0 \sum_{j=1}^N \eta_j \leq \beta^N \varepsilon$$

if $\varepsilon \leq \varepsilon_1$. This proves (3.6). In a similar way, (3.14) implies (3.7). Finally if $w(0) \in \hat{L}^\alpha$, we use (3.7) to obtain

$$\begin{aligned} \|w\|_{L^\infty(I, \hat{L}^\alpha)} &\leq \|w(0)\|_{\hat{L}^\alpha} + C \| |u|^{2\alpha} u - |\tilde{u}|^{2\alpha} \tilde{u} \|_{N(I)} + C \|e\|_{N(I)} \\ &\leq \|w(0)\|_{\hat{L}^\alpha} + C\varepsilon \end{aligned}$$

This completes the proof of Proposition 3.2. \square

3.2. A version of small data scattering. As a simple consequence of Proposition 3.2, we have the following result, which is Theorem 1.6.

Corollary 3.3. *Let $5/3 \leq \alpha < 20/9$. For any $M > 0$ there exists $\delta = \delta(M) > 0$ such that if $u_0 \in \hat{L}^\alpha$ satisfies $\|u_0\|_{\hat{L}^\alpha} \leq M$ and*

$$\varepsilon := \left\| e^{-t\partial_x^3} u_0 \right\|_{L(\mathbb{R})} \leq \delta$$

then a corresponding solution $u(t)$ to (gKdV) exists globally and scatters for both time directions. Further, it holds that

$$\|u\|_{S(\mathbb{R})} + \|u\|_{L(\mathbb{R})} \leq M + CM^{2\alpha}\varepsilon$$

for some constant C .

Proof. We just apply Proposition 3.2 with $\tilde{u}(t, x) = e^{-t\partial_x^3} u_0$, $I = \mathbb{R}$, and $\hat{t} = 0$. Remark that

$$\|\tilde{u}\|_{S(\mathbb{R})} + \|\tilde{u}\|_{L(\mathbb{R})} \leq C \|u_0\|_{\hat{L}^\alpha} \leq CM$$

follows from (2.4) and by assumption. Further, $u(0) - \tilde{u}(0) \equiv 0$ and

$$\|e\|_{N(\mathbb{R})} = \left\| |\tilde{u}|^{2\alpha} \tilde{u} \right\|_{N(\mathbb{R})} \leq C \|\tilde{u}\|_{S(\mathbb{R})}^{2\alpha} \|\tilde{u}\|_{L(\mathbb{R})} \leq CM^{2\alpha}\varepsilon \leq \varepsilon_1$$

for sufficiently small $\delta = \delta(M)$, where ε_1 is the constant given in Proposition 3.2. Therefore, the assumption of Proposition 3.2 is satisfied. \square

4. PROOF OF MAIN THEOREMS

4.1. Two tools. For the proof of Theorem 1.2, we introduce the following two tools.

The first one is a linear profile decomposition for \hat{L}^α -bounded sequences. Let us define a set of deformations as follows

$$(4.1) \quad G := \{D(h)A(s)T(y)P(\xi) \mid \Gamma = (h, \xi, s, y) \in 2^{\mathbb{Z}} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}\}.$$

We often identify $\mathcal{G} \in G$ with a corresponding parameter $\Gamma \in 2^{\mathbb{Z}} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ if there is no fear of confusion. Let us now introduce a notion of orthogonality between two families of deformations.

Definition 4.1. *We say two families of deformations $\{\mathcal{G}_n\} \subset G$ and $\{\tilde{\mathcal{G}}_n\} \subset G$ are orthogonal if corresponding parameters $\Gamma_n, \tilde{\Gamma}_n \in 2^{\mathbb{Z}} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ satisfies*

$$(4.2) \quad \lim_{n \rightarrow \infty} \left(\left| \log \frac{h_n}{\tilde{h}_n} \right| + \left| \xi_n - \frac{\tilde{h}_n}{h_n} \tilde{\xi}_n \right| + \left| s_n - \left(\frac{h_n}{\tilde{h}_n} \right)^3 \tilde{s}_n \right| (1 + |\xi_n|) \right. \\ \left. + \left| y_n - \frac{h_n}{\tilde{h}_n} \tilde{y}_n - 3 \left(s_n - \left(\frac{h_n}{\tilde{h}_n} \right)^3 \tilde{s}_n \right) (\xi_n)^2 \right| \right) = +\infty.$$

Remark 4.2. It follows from (2.1) that

$$\begin{aligned} (\tilde{\mathcal{G}}_n)^{-1} \mathcal{G}_n &= e^{i\gamma_n} D \left(\frac{h_n}{\tilde{h}_n} \right) P \left(\xi_n - \frac{\tilde{h}_n}{h_n} \tilde{\xi}_n \right) \\ &\quad A \left(s_n - \left(\frac{h_n}{\tilde{h}_n} \right)^3 \tilde{s}_n \right) S \left(3 \left(s_n - \left(\frac{h_n}{\tilde{h}_n} \right)^3 \tilde{s}_n \right) \xi_n \right) \\ &\quad T \left(y_n - \frac{h_n}{\tilde{h}_n} \tilde{y}_n - 3 \left(s_n - \left(\frac{h_n}{\tilde{h}_n} \right)^3 \tilde{s}_n \right) \xi_n^2 \right), \end{aligned}$$

where Γ_n and $\tilde{\Gamma}_n$ are parameters associated with \mathcal{G}_n and $\tilde{\mathcal{G}}_n$, respectively, and γ_n is a real constant given by Γ_n and $\tilde{\Gamma}_n$. Intuitively, the orthogonality given in Definition 4.1 implies at least one of the deformations in the right hand side produces bad behavior.

Theorem 4.3 (Linear profile decomposition for “real valued” functions). *Let $4/3 < \alpha < 2$ and $\alpha' < \sigma < \frac{6\alpha}{3\alpha-2}$. Let $u = \{u_n\}_n$ be a sequence of real-valued functions in B_M . Then, there exist $\psi^j \in B_M$, $r_n^j \in B_{(2j+1)M}$ and pairwise orthogonal families of deformations $\{\mathcal{G}_n^j\}_n \subset G$ ($j = 1, 2, \dots$) parametrized by $\{\Gamma_n^j = (h_n^j, \xi_n^j, s_n^j, y_n^j)\}_n$ such that, extracting a subsequence in n ,*

$$(4.3) \quad u_n = \sum_{j=1}^l \text{Re}(\mathcal{G}_n^j \psi^j) + r_n^l$$

for all $l \geq 1$ and

$$(4.4) \quad \limsup_{n \rightarrow \infty} \left\| e^{-t\partial_x^3} r_n^l \right\|_{L(\mathbb{R})} \rightarrow 0$$

as $l \rightarrow \infty$. For all $j \geq 1$,

$$\text{either } \xi_n^j = 0, \forall n \geq 0 \quad \text{or} \quad \xi_n^j \rightarrow \infty \text{ as } n \rightarrow \infty.$$

Moreover, a decoupling inequality

$$(4.5) \quad \limsup_{n \rightarrow \infty} \ell(u_n) \geq \left(\sum_{j=1}^J c_j^{1-\sigma} \ell(\psi^j)^\sigma \right)^{1/\sigma} + \limsup_{n \rightarrow \infty} \ell(r_n^J)$$

holds for all $J \geq 1$, where

$$c_j = \begin{cases} 1 & \text{if } \xi_n^j \equiv 0, \\ 2 & \text{if } \xi_n^j \rightarrow \infty \text{ as } n \rightarrow \infty. \end{cases}$$

Furthermore, it holds that

$$(4.6) \quad c_j \|\psi^j\|_{\dot{L}^\alpha} \leq \limsup_{n \rightarrow \infty} \|u_n\|_{\dot{L}^\alpha}$$

for any j .

The second tool to prove Theorem 1.2 is uniform boundedness of solutions with highly oscillating initial data. The assumption (1.8) is necessary for this boundedness.

Theorem 4.4. *Let $12/7 < \alpha < 2$. Assume (1.8). Let $\phi \in \hat{L}_x^\alpha(\mathbb{R})$ be a complex valued function such that*

$$\ell(\phi) < 2^{1-\frac{1}{\sigma}} d_+.$$

Let $\{\xi_n\}_{n \geq 1} \subset (0, \infty)$ with $\xi_n \rightarrow \infty$ and let $\{t_n\}_{n \geq 1} \subset \mathbb{R}$ be such that $-3t_n\xi_n$ converges to some $T_0 \subset [-\infty, \infty]$. Then for n sufficiently large, a corresponding \hat{L}^α -solution u_n to (gKdV) with the initial condition

$$(4.7) \quad u_n(t_n, x) = A(t_n) \operatorname{Re}[P(\xi_n)\phi(x)]$$

exists globally in time. Moreover, the solution u_n satisfies a uniform space-time bound $\|u_n\|_{S(\mathbb{R})} + \|u_n\|_{L(\mathbb{R})} \leq C$, where C is a positive constant depending only on ϕ .

We postpone the proof of Theorems 4.3 and 4.4 to Sections 5 and 8, respectively.

4.2. Proof of Theorem 1.2.

Step 1. Take a minimizing sequence $\{u_n\}_n$ as follows;

$$(4.8) \quad u_n \in B_M \setminus \mathcal{S}_+, \quad \ell(u_n) \leq d_+ + \frac{1}{n}.$$

We apply the linear profile decomposition theorem (Theorem 4.3) to the sequence $\{u_n\}_n$. Then, up to subsequence, we obtain a decomposition

$$(4.9) \quad u_n = \sum_{j=1}^l \operatorname{Re}(\mathcal{G}_n^j \psi^j) + r_n^l$$

for $n, l \geq 1$. By extracting subsequence and changing notations if necessary, we may assume that for each j and $\{x_n^j\}_{n,j} = \{\log h_n^j\}_{n,j}, \{t_n^j\}_{n,j}, \{y_n^j\}_{n,j}, \{3\xi_n^j t_n^j\}$, either $x_n^j \equiv 0$, $x_n^j \rightarrow \infty$ as $n \rightarrow \infty$, or $x_n^j \rightarrow -\infty$ as $n \rightarrow \infty$ holds.

Step 2. In this step and the next step, we shall show that $\psi^j \equiv 0$ except for at most one j .

Suppose not. Then, by means of (4.5), we have $c_j^{\frac{1}{\sigma}-1} \ell(\psi^j) < d_+$ for all j .

Let us define $V_n^j(t, x)$ as follows:

- When $\xi_n \equiv 0$, we let $V_n^j(t) = D(h_n^j)T(y_n^j)\Psi^j((h_n^j)^3 t + t_n^j)$, where $\Psi^j(t)$ is a nonlinear profile associated with $(\operatorname{Re} \psi^j, t_n^j)$, that is,
 - if $t_n^j \equiv 0$ then $\Psi^j(t)$ is a solution to (gKdV) with $\Psi^j(0) = \operatorname{Re} \psi^j$;
 - if $t_n^j \rightarrow \infty$ as $n \rightarrow \infty$ then $\Psi^j(t)$ is a solution to (gKdV) that scatters forward in time to $e^{-t\partial_x^3} \operatorname{Re} \psi^j$;
 - if $t_n^j \rightarrow -\infty$ as $n \rightarrow \infty$ then $\Psi^j(t)$ is a solution to (gKdV) that scatters backward in time to $e^{-t\partial_x^3} \operatorname{Re} \psi^j$;
- When $\xi_n \rightarrow \infty$, we let $V_n^j(t) = D(h_n^j)T(y_n^j)\Psi_n^j((h_n^j)^3 t + t_n^j)$, where Ψ_n^j is a solution to (gKdV) with the initial condition

$$\Psi_n^j(t_n^j) = A(t_n^j) \operatorname{Re}(P(\xi_n^j)\psi^j).$$

Let us show the following two lemmas.

Lemma 4.5 (uniform bound on the approximate solution). *There exists $M > 0$ such that*

$$\|V_n^j\|_{K(\mathbb{R}_+)} + \|V_n^j\|_{Z(\mathbb{R}_+)} \leq M$$

holds for any $j, n \geq 1$.

Proof. The case $\xi_n^j \rightarrow \infty$ follows from Theorem 4.4. Hence, here we assume that $\xi_n^j \equiv 0$. Note that $c_j = 1$. Since the deformations $D(h_n^j)$ and $T(y_n^j)$ leave the left hand side invariant, it suffices to show that

$$\|\Psi^j\|_{K((t_n^j, \infty))} + \|\Psi^j\|_{Z((t_n^j, \infty))}$$

is bounded uniformly in n . Since $\ell(\psi^j) < d_+$ by assumption, Ψ^j scatters forward in time. Hence, if $t_n^j \equiv 0$ or if $t_n^j \rightarrow \infty$ as $n \rightarrow \infty$ then

$$\|\Psi^j\|_{K((t_n^j, \infty))} + \|\Psi^j\|_{Z((t_n^j, \infty))} \leq \|\Psi^j\|_{K((0, \infty))} + \|\Psi^j\|_{Z((0, \infty))} < \infty$$

by scattering criterion. If $t_n \rightarrow -\infty$ as $n \rightarrow \infty$ then Ψ^j scatters for both time directions and so

$$\|\Psi^j\|_{K((t_n^j, \infty))} + \|\Psi^j\|_{Z((t_n^j, \infty))} \leq \|\Psi^j\|_{K(\mathbb{R})} + \|\Psi^j\|_{Z(\mathbb{R})} < \infty.$$

Hence, we obtain Lemma 4.5. \square

Next lemma is concerned with the decoupling of the nonlinear profile.

Lemma 4.6. *For any $j \neq k$, we have*

$$\lim_{n \rightarrow \infty} \left\| V_n^j V_n^k \right\|_{L_x^{\frac{p(S)}{2}} L_t^{\frac{q(S)}{2}}(\mathbb{R})} = 0,$$

where $(p(S), q(S)) = (\frac{5}{2}\alpha, 5\alpha)$.

To prove Lemma 4.6, it suffices to show the following lemma.

Lemma 4.7. *Set*

$$\tilde{V}_n^j := \begin{cases} D(h_n^j)T(y_n^j)\Psi^j((h_n^j)^3t + t_n^j, x) & (\text{if } \xi_n^j \equiv 0), \\ D(h_n^j)T(y_n^j)\Psi^j(-3(\xi_n^j)[(h_n^j)^3t + t_n^j], x + 3(\xi_n^j)^2[(h_n^j)^3t + t_n^j]) & (\text{if } \xi_n^j \rightarrow \infty \text{ as } n \rightarrow \infty). \end{cases}$$

Then for any $j \neq k$, we have

$$(4.10) \quad \lim_{n \rightarrow \infty} \left\| \tilde{V}_n^j \tilde{V}_n^k \right\|_{L_x^{\frac{p(S)}{2}} L_t^{\frac{q(S)}{2}}(\mathbb{R})} = 0.$$

Proof. The proof is now standard. Let us first prove (4.10) when $\xi_n^j \rightarrow \infty$ and $\xi_n^k \rightarrow \infty$ as $n \rightarrow \infty$. By density, it suffices to handle the case $\Psi^j(t, x) \equiv \Psi^k(t, x) \equiv \mathbf{1}_{\{|t| \leq C, |x| \leq C\}}(t, x)$, where $C > 0$. Note that

$$\begin{aligned} & D(h_n^j)T(y_n^j)\Psi^j(-3(\xi_n^j)[(h_n^j)^3t + t_n^j], x + 3(\xi_n^j)^2[(h_n^j)^3t + t_n^j]) \\ &= (h_n^j)^{\frac{1}{\alpha}} \Psi^j(-3(\xi_n^j)[(h_n^j)^3t + t_n^j], (h_n^j)x - y_n^j + 3(\xi_n^j)^2[(h_n^j)^3t + t_n^j]). \end{aligned}$$

Case 1: $\lim_{n \rightarrow \infty} |\log \frac{h_n^j}{h_n^k}| = \infty$. We may assume $\lim_{n \rightarrow \infty} \frac{h_n^k}{h_n^j} = \infty$. The Hölder inequality yields

$$\left\| \tilde{V}_n^j \tilde{V}_n^k \right\|_{L_x^{\frac{p(S)}{2}} L_t^{\frac{q(S)}{2}}(\mathbb{R})}^{\frac{5\alpha}{4}} \leq (h_n^j)^{\frac{5}{4}} (h_n^k)^{\frac{5}{4}} \|\mathbf{1}_{R_n^j}\|_{L_x^\infty L_t^\infty}^{\frac{5\alpha}{4}} \|\mathbf{1}_{R_n^k}\|_{L_x^{\frac{5\alpha}{4}} L_t^{\frac{5\alpha}{2}}}^{\frac{5\alpha}{4}}$$

$$\leq C(h_n^j)^{\frac{5}{4}}(h_n^k)^{\frac{5}{4}}|I_n^k|^{\frac{1}{2}}|\text{Area}R_n^k|^{\frac{1}{2}},$$

where

$$\begin{aligned} I_n^k &= \{x \in \mathbb{R} \mid \|\mathbf{1}_{R_n^k}(\cdot, x)\|_{L_t^1} \neq 0\}, \\ R_n^k &= \{(t, x) \in \mathbb{R}^2 \mid 3(\xi_n^k)|h_n^k|^3 t + t_n^k| \leq C, \\ &\quad |(h_n^k)x - y_n^k + 3(\xi_n^k)^2[(h_n^k)^3 t + t_n^k]| \leq C\}. \end{aligned}$$

We easily see that

$$|I_n^k| \leq C \frac{\langle \xi_n^k \rangle}{h_n^k}, \quad |\text{Area}R_n^k| \leq \frac{C}{(h_n^k)^4 \xi_n^k}.$$

Since $\xi_n^k \rightarrow \infty$ as $n \rightarrow \infty$, we have

$$\left\| \tilde{V}_n^j \tilde{V}_n^k \right\|_{L_x^{\frac{p(S)}{2}} L_t^{\frac{q(S)}{2}}(\mathbb{R})}^{\frac{5\alpha}{4}} \leq C \left(\frac{h_n^j}{h_n^k} \right)^{\frac{5}{4}} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Case 2: $|\log \frac{h_n^j}{h_n^k}| < \infty$ and $\lim_{n \rightarrow \infty} |\xi_n^j - \frac{h_n^k}{h_n^j} \xi_n^k| = \infty$. The Hölder inequality yields

$$(4.11) \quad \left\| \tilde{V}_n^j \tilde{V}_n^k \right\|_{L_x^{\frac{p(S)}{2}} L_t^{\frac{q(S)}{2}}(\mathbb{R})}^{\frac{5\alpha}{4}} \leq C(h_n^j)^{\frac{5}{4}}(h_n^k)^{\frac{5}{4}}|I_n^{j,k}|^{\frac{1}{2}}|\text{Area}(R_n^j \cap R_n^k)|^{\frac{1}{2}},$$

where $I_n^{j,k} = \{x \in \mathbb{R} \mid \|\mathbf{1}_{R_n^j} \mathbf{1}_{R_n^k}\|_{L_t^1} \neq 0\}$. Since $R_n^j \subset \{(t, x) \in \mathbb{R}^2 \mid |(h_n^j)x - y_n^j + 3(\xi_n^j)^2[(h_n^j)^3 t + t_n^j]| \leq C\}$, changing variables $(t, x) \mapsto (v, w)$:

$$\begin{aligned} v &= (h_n^j)x - y_n^j + 3(\xi_n^j)^2[(h_n^j)^3 t + t_n^j], \\ w &= (h_n^k)x - y_n^k + 3(\xi_n^k)^2[(h_n^k)^3 t + t_n^k], \end{aligned}$$

we have

$$\begin{aligned} (4.12) \quad \text{Area}(R_n^j \cap R_n^k) &\leq \int_{|v| \leq C} \int_{|w| \leq C} \left| \frac{\partial(t, x)}{\partial(v, w)} \right| dv dw \\ &\leq \frac{C}{h_n^j h_n^k |(\xi_n^j)^2 (h_n^j)^2 - (\xi_n^k)^2 (h_n^k)^2|}. \end{aligned}$$

Furthermore, we easily see

$$(4.13) \quad |I_n^{j,k}| \leq C \min \left\{ \frac{\xi_n^j}{h_n^j}, \frac{\xi_n^k}{h_n^k} \right\}.$$

Plugging (4.12) and (4.13) into (4.11), we have

$$\left\| \tilde{V}_n^j \tilde{V}_n^k \right\|_{L_x^{\frac{p(S)}{2}} L_t^{\frac{q(S)}{2}}(\mathbb{R})}^{\frac{5\alpha}{4}} \leq \frac{C}{\left| \xi_n^j - \frac{h_n^k}{h_n^j} \xi_n^k \right|^2} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

where we have used that $h_n^j \sim h_n^k$.

Case 3: $|\log \frac{h_n^j}{h_n^k}| + |\xi_n^j - \frac{h_n^k}{h_n^j} \xi_n^k| < \infty$ and $\lim_{n \rightarrow \infty} |t_n^j - (\frac{h_n^j}{h_n^k})^3 t_n^k| (1 + |\xi_n^j|) = \infty$.

Let $T_n^j = \{t \in \mathbb{R} \mid \exists x \in \mathbb{R} \text{ such that } (t, x) \in R_n^j\}$. We see that for n sufficiently

large, $T_n^j \cap T_n^k = \phi$. Indeed, we have

$$T_n^j \cap T_n^k = \phi \Leftrightarrow \left| t_n^j - \left(\frac{h_n^j}{h_n^k} \right)^3 t_n^k \right| |\xi_n^j| > C$$

for some large positive constant. Since the right hand side of the above inequality goes to infinity as $n \rightarrow \infty$, we have that for n sufficiently large, $T_n^j \cap T_n^k = \phi$. Therefore the Hölder inequality yields (4.10).

Case 4: $|\log \frac{h_n^j}{h_n^k}| + |\xi_n^j - \frac{h_n^k}{h_n^j} \xi_n^k| + |t_n^j - (\frac{h_n^j}{h_n^k})^3 t_n^k| (1 + |\xi_n^j|) < \infty$ and $\lim_{n \rightarrow \infty} |y_n^j - \frac{h_n^j}{h_n^k} y_n^k - 3[t_n^j - (\frac{h_n^j}{h_n^k})^3 t_n^k](\xi_n^j)^2| = \infty$. Let $X_n^j(t) = \{x \in \mathbb{R} | (t, x) \in R_n^j\}$. We see that if $t \in T_n^j \cap T_n^k$ for n sufficiently large, then $X_n^j(t) \cap X_n^k(t) = \phi$. Indeed, we have

$$X_n^j(t) \cap X_n^k(t) = \phi \Leftrightarrow \left| y_n^j - \frac{h_n^j}{h_n^k} y_n^k - 3 \left\{ t_n^j - \frac{(h_n^j)^3}{(h_n^k)^3} t_n^k \right\} (\xi_n^j)^2 \right| > C$$

for some large positive constant. Since the right hand side of the above inequality goes to infinity as $n \rightarrow \infty$, we have that for n sufficiently large, $X_n^j(t) \cap X_n^k(t) = \phi$. Therefore the Hölder inequality yields (4.10).

By arguments similar to those in cases 1, 3, and 4, we obtain (4.10) when $\xi_n^j \equiv 0$ and $\xi_n^k \rightarrow \infty$ as $n \rightarrow \infty$. Similarly, arguing as in cases 1 and 2, we obtain (4.10) when $\xi_n^j \equiv \xi_n^k \equiv 0$ as $n \rightarrow \infty$. This completes the proof of Lemma 4.7. \square

Lemma 4.8. *Let $F(z) = |z|^{2\alpha} z$. For any $\mathcal{J} \subset \mathbb{Z}_+$,*

$$\left\| F \left(\sum_{j \in \mathcal{J}} V_n^j \right) - \sum_{j \in \mathcal{J}} F(V_n^j) \right\|_{L_x^{\frac{p(S)}{2\alpha+1}} L_t^{\frac{q(S)}{2\alpha+1}}(\mathbb{R}_+)} = o(1)$$

as $n \rightarrow \infty$. Similarly,

$$\left\| F \left(\sum_{j \in \mathcal{J}} V_n^j \right) - \sum_{j \in \mathcal{J}} F(V_n^j) \right\|_{N(\mathbb{R}_+)} = o(1)$$

as $n \rightarrow \infty$.

Proof. The former estimate is a consequence of Lemma 4.6. Indeed, we see that

$$\begin{aligned} \left| F \left(\sum_{j \in \mathcal{J}} V_n^j \right) - \sum_{j \in \mathcal{J}} F(V_n^j) \right| &\leq C \left| \sum_{j_1, j_2 \in \mathcal{J}, j_1 \neq j_2} V_n^{j_1} V_n^{j_2} \right| \\ &\quad \left| \sum_{j_3, j_4 \in \mathcal{J}, j_3 \neq j_4} V_n^{j_3} V_n^{j_4} \right| \left| \sum_{j_5, j_6 \in \mathcal{J}, j_5 \neq j_6} V_n^{j_5} V_n^{j_6} \right|^{\frac{2\alpha-3}{2}}. \end{aligned}$$

Therefore, the Hölder inequality and Lemma 4.6 give us the desired estimate. Take a conjugate-acceptable pair $(s(N'), \alpha)$ so that $0 < s(N') - s(N) \ll 1$,

and set $N'(I) := Y(I; s(N'), \alpha)$. By means of the interpolation estimate (Lemma 2.8), the latter estimate follows if we show that

$$\left\| \left(F \left(\sum_{j \in \mathcal{J}} V_n^j \right) - \sum_{j \in \mathcal{J}} F(V_n^j) \right) \right\|_{N'(\mathbb{R}_+)}$$

is bounded uniformly in n . When $s(N')$ is chosen sufficiently close to $s(N)$, we have

$$\|F(u)\|_{\mathbb{R}_+} \leq C \|u\|_{L(\mathbb{R}_+) \cap S(\mathbb{R}_+)}^{2\alpha+1}$$

just as in the proof of Lemma 2.9. Therefore,

$$\begin{aligned} & \left\| \left(F \left(\sum_{j \in \mathcal{J}} V_n^j \right) - \sum_{j \in \mathcal{J}} F(V_n^j) \right) \right\|_{N'(\mathbb{R}_+)} \\ & \leq \left\| F \left(\sum_{j \in \mathcal{J}} V_n^j \right) \right\|_{N'(\mathbb{R}_+)} + \sum_{j \in \mathcal{J}} \|F(V_n^j)\|_{N'(\mathbb{R}_+)} \\ & \leq C \left\| \sum_{j \in \mathcal{J}} V_n^j \right\|_{L(\mathbb{R}_+) \cap S(\mathbb{R}_+)}^{2\alpha+1} + C \sum_{j \in \mathcal{J}} \|V_n^j\|_{L(\mathbb{R}_+) \cap S(\mathbb{R}_+)}^{2\alpha+1} \\ & \leq C \sum_{j \in \mathcal{J}} \|V_n^j\|_{L(\mathbb{R}_+) \cap S(\mathbb{R}_+)}^{2\alpha+1}. \end{aligned}$$

The right hand side is bounded uniformly in n , thanks to Lemma 4.8, which completes the proof. \square

Step 3. Here, we define an approximate solution

$$(4.14) \quad \tilde{u}_n^J(t, x) = \sum_{j=1}^J V_n^j(t, x) + e^{-t\partial_x^3} r_n^J,$$

where V_n^j is given in Step 2. To apply long time stability, we now check that \tilde{u}_n^J satisfies the assumption.

Proposition 4.9 (Asymptotic agreement at the initial time). *Let \tilde{u}_n^J and u_n be given by (4.14) and (4.8), respectively. Then it holds for any $J \geq 1$ that*

$$\|\tilde{u}_n^J(0) - u_n\|_{\hat{L}^\alpha} \rightarrow 0$$

as $n \rightarrow \infty$.

Proof. This follows from

$$V_n^j(0) - \mathcal{G}_n^j \psi^j \rightarrow 0 \quad \text{in } \hat{L}^\alpha$$

for each j , which is an immediate consequence of the way V_n^j are constructed. \square

Proposition 4.10 (Uniform bound on the approximate solution). *There exists $M > 0$ such that*

$$\|\tilde{u}_n^J\|_{K(\mathbb{R}_+)} + \|\tilde{u}_n^J\|_{Z(\mathbb{R}_+)} \leq M$$

holds for any $J \geq 1$ and $n \geq N(J)$.

Recall that each V_n^j ($j \geq 1$) is bounded in $K(0, \infty) \cap Z(0, \infty)$ uniformly in n (Lemma 4.5). Further, $e^{-t\partial_x^3} r_n^J$ is also bounded uniformly in $J, n \geq 1$. Hence, we shall show that there exists J_0 such that

$$\left\| \sum_{j=J_0+1}^{J_0+k} V_n^j \right\|_{K(\mathbb{R}_+)} + \left\| \sum_{j=J_0+1}^{J_0+k} V_n^j \right\|_{Z(\mathbb{R}_+)} \leq C$$

for any $k \geq 1$ and $n \geq N(k)$. To this end, we need the following.

Lemma 4.11. *For any $\varepsilon > 0$, there exists $J_0 = J_0(\varepsilon)$ such that*

$$\left\| \sum_{j=J_0+1}^{J_0+k} e^{-t\partial_x^3} V_n^j(0) \right\|_{L(\mathbb{R}_+)} + \left\| \sum_{j=J_0+1}^{J_0+k} e^{-t\partial_x^3} V_n^j(0) \right\|_{S(\mathbb{R}_+)} \leq \varepsilon$$

for any $k \geq 1$ and $n \geq N(k)$.

Proof. By Proposition 4.9, it suffices to prove the estimate for $e^{-t\partial_x^3} \mathcal{G}_n^j \psi^j$ instead of $e^{-t\partial_x^3} V_n^j(0)$. By Theorem 6.5 in Section 6, we see that

$$\begin{aligned} \left\| \sum_{j=J_0+1}^{J_0+k} e^{-t\partial_x^3} \mathcal{G}_n^j \psi^j \right\|_{L(\mathbb{R}_+)} &\leq C \left\| \sum_{j=J_0+1}^{J_0+k} \mathcal{G}_n^j \psi^j \right\|_{\dot{M}_{2,\sigma}^\alpha} \\ &\leq C \left(\sum_{j=J_0+1}^{J_0+k} \|\psi^j\|_{\dot{M}_{2,\sigma}^\alpha}^\sigma \right)^{1/\sigma} + o(1). \end{aligned}$$

Therefore, for any $\varepsilon > 0$, we can choose $J_0(\varepsilon)$ so that

$$\left\| \sum_{j=J_0+1}^{J_0+k} e^{-t\partial_x^3} \mathcal{G}_n^j \psi^j \right\|_{L(\mathbb{R}_+)} \leq \frac{\varepsilon}{2}$$

for any $k \geq 1$ and $n \geq n(k)$.

On the other hand,

$$\left\| \sum_{j=J_0+1}^{J_0+k} e^{-t\partial_x^3} \mathcal{G}_n^j \psi^j \right\|_{S(\mathbb{R}_+)}^{p(S)} \leq \sum_{j=J_0+1}^{J_0+k} \left\| e^{-t\partial_x^3} \mathcal{G}_n^j \psi^j \right\|_{S(\mathbb{R}_+)}^{p(S)} + o(1)$$

as $n \rightarrow \infty$. Since

$$\begin{aligned} \left\| e^{-t\partial_x^3} \mathcal{G}_n^j \psi^j \right\|_{S(\mathbb{R}_+)} &\leq C \left\| e^{-t\partial_x^3} \mathcal{G}_n^j \psi^j \right\|_{L(\mathbb{R}_+)}^\theta \left\| e^{-t\partial_x^3} \mathcal{G}_n^j \psi^j \right\|_{Z(\mathbb{R}_+)}^{1-\theta}, \\ &\leq C \left\| \mathcal{G}_n^j \psi^j \right\|_{\dot{M}_{2,\sigma}^\alpha}^\theta \left\| \psi^j \right\|_{\dot{L}^\alpha}^{1-\theta} \\ &\leq C \left\| \mathcal{G}_n^j \psi^j \right\|_{\dot{M}_{2,\sigma}^\alpha}^\theta, \end{aligned}$$

where $\theta = -s(Z)/(s(L) - s(Z))$, one verifies that

$$\sum_{j=J_0+1}^{J_0+k} \left\| e^{-t\partial_x^3} \mathcal{G}_n^j \psi^j \right\|_{S(\mathbb{R}_+)}^{p(S)} \leq C \sum_{j=J_0+1}^{\infty} \left\| \mathcal{G}_n^j \psi^j \right\|_{\dot{M}_{2,\sigma}^{\alpha}}^{\theta p(S)}.$$

The right hand side is bounded since $\theta p(S) > \sigma$. Hence, we can choose $J_1(\varepsilon)$ so that

$$\sum_{j=J_1+1}^{J_1+k} \left\| e^{-t\partial_x^3} \mathcal{G}_n^j \psi^j \right\|_{S(\mathbb{R}_+)} \leq \frac{\varepsilon}{2}$$

for any $k \geq 1$ and $n \geq n(k)$. \square

Remark 4.12. Our assumption $\alpha > 3/2 + \sqrt{7/60}$ comes from the condition $\theta p(S) > \sigma$ in this lemma. By letting $s(Z) \downarrow \frac{2}{\alpha} - \frac{5}{4}$, we have $\theta p(S) \rightarrow \frac{3\alpha(5\alpha-8)}{2(3\alpha-4)}$. This upper bound of σ , which restrict us to the above range of α with lower bound $\sigma > \alpha'$, is used only in this lemma, and all other arguments work with a weaker assumption $\sigma < 6\alpha/(3\alpha - 2)$. Here, we also remark on the choice of the space $Z(I)$. For any fixed $\alpha > 3/2 + \sqrt{7/60}$, we are able to choose σ so that $\alpha' < \sigma < \frac{3\alpha(5\alpha-8)}{2(3\alpha-4)}$. Then, we fix the space $Z(I)$ so that $\theta = -s(Z)/(s(L) - s(Z))$ satisfies $\theta p(S) > \sigma$.

We now prove Proposition 4.10.

Proof. The integral equation that $W_n^k := \sum_{j=J_0+1}^{J_0+k} V_n^j$ satisfies is

$$W_n^k = e^{-t\partial_x^3} W_n^k(0) + \mu \int_0^t e^{-(t-s)\partial_x^3} \partial_x (|W_n^k|^{2\alpha} W_n^k + E_n^k) ds,$$

where

$$-E_n^k = |W_n^k|^{2\alpha} W_n^k - \sum_{j=J_0+1}^{J_0+k} |V_n^j|^{2\alpha} V_n^j.$$

Therefore,

$$\begin{aligned} \left\| W_n^k \right\|_{L(\mathbb{R}_+) \cap S(\mathbb{R}_+)} &\leq \left\| e^{-t\partial_x^3} W_n^k(0) \right\|_{L(\mathbb{R}_+) \cap S(\mathbb{R}_+)} + C \left\| W_n^k \right\|_{L(\mathbb{R}_+) \cap S(\mathbb{R}_+)}^{2\alpha+1} \\ &\quad + C \|E_n\|_{N(\mathbb{R}_+)}. \end{aligned}$$

Fix $\varepsilon > 0$. Thanks to Lemma 4.11, one can choose J_0 so that

$$\left\| e^{-t\partial_x^3} W_n^k(0) \right\|_{L(\mathbb{R}_+) \cap S(\mathbb{R}_+)} \leq \varepsilon$$

for any $k \geq 1$ and $n \geq N(k)$. Further, for this J_0 , we have

$$\|E_n\|_{N(\mathbb{R}_+)} \leq \varepsilon$$

for any $k \geq 1$ and $n \geq N(k)$. \square

Proposition 4.13 (Approximate solution to the equation). *Let \tilde{u}_n^J be defined by (4.14). Then \tilde{u}_n^J is an approximate solution to (gKdV) in such a sense that*

$$\lim_{J \rightarrow \infty} \limsup_{n \rightarrow \infty} \left\| |\partial_x|^{-1} [(\partial_t + \partial_{xxx}) \tilde{u}_n^J - \mu \partial_x (|\tilde{u}_n^J|^{2\alpha} \tilde{u}_n^J)] \right\|_{N(\mathbb{R}_+)} = 0.$$

Proof. First note that the identity

$$\begin{aligned}
& (\partial_t + \partial_{xxx})\tilde{u}_n^J - \mu\partial_x(|\tilde{u}_n^J|^{2\alpha}\tilde{u}_n^J) \\
&= \mu \sum_{j=1}^J \partial_x(|V_n^j|^{2\alpha}V_n^j) - \mu\partial_x(|\tilde{u}_n^J|^{2\alpha}\tilde{u}_n^J) \\
&= \mu\partial_x \left\{ \sum_{j=1}^J (|V_n^j|^{2\alpha}V_n^j) - \left(\left| \sum_{j=1}^J V_n^j \right|^{2\alpha} \sum_{j=1}^J V_n^j \right) \right\} \\
&\quad + \mu\partial_x \left\{ \left(\left| \sum_{j=1}^J V_n^j \right|^{2\alpha} \sum_{j=1}^J V_n^j \right) - (|\tilde{u}_n^J|^{2\alpha}\tilde{u}_n^J) \right\} \\
&=: \partial_x I_1 + \partial_x I_2.
\end{aligned}$$

Lemma 4.8 implies $\lim_{n \rightarrow \infty} \|I_1\|_{N(\mathbb{R}_+)} = 0$. Therefore, we only have to handle I_2 . From Lemma 2.9 (ii) and Proposition 4.10, we have

$$\begin{aligned}
& \|I_2\|_{N(\mathbb{R}_+)} \\
&\leq C(\|u_n\|_{L(\mathbb{R}_+) \cap S(\mathbb{R}_+)} + \|e^{-t\partial_x^3}r_n^J\|_{L(\mathbb{R}_+) \cap S(\mathbb{R}_+)})^{2\alpha} \|e^{-t\partial_x^3}r_n^J\|_{L(\mathbb{R}_+) \cap S(\mathbb{R}_+)} \\
&\leq C \|e^{-t\partial_x^3}r_n^J\|_{L(\mathbb{R}_+) \cap S(\mathbb{R}_+)}
\end{aligned}$$

for any $n \geq N(J)$. By (4.4),

$$\lim_{J \rightarrow \infty} \limsup_{n \rightarrow \infty} \|e^{-t\partial_x^3}r_n^J\|_L = 0$$

and

$$\begin{aligned}
\limsup_{n \rightarrow \infty} \|e^{-t\partial_x^3}r_n^J\|_{S(\mathbb{R}_+)} &\leq C \limsup_{n \rightarrow \infty} \|e^{-t\partial_x^3}r_n^J\|_{L(\mathbb{R}_+)}^\theta \|e^{-t\partial_x^3}r_n^J\|_{Z(\mathbb{R}_+)}^{1-\theta} \\
&\leq CM^{1-\theta} \limsup_{n \rightarrow \infty} \|e^{-t\partial_x^3}r_n^J\|_{L(\mathbb{R}_+)}^\theta \rightarrow 0
\end{aligned}$$

as $J \rightarrow \infty$. This yields $\lim_{n \rightarrow \infty} \|I_2\|_{N(\mathbb{R}_+)} = 0$. Hence we have the desired estimate. \square

Now, we apply long time stability to see that $\|u_n\|_{S(\mathbb{R}_+)} < \infty$ for sufficiently large n . This implies that $u_n \in \mathcal{S}_+$, which contradicts with the definition of $\{u_n\}_n$.

Step 4. We now see that there exists j_0 such that $c_j^{\frac{1}{\sigma}-1} \ell(\psi^{j_0}) = d_{+, \text{gKdV}}$. Then, one sees from the definition of $\{u_n\}_n$ and (4.5) that $\psi^j \equiv 0$ for $j \neq j_0$. For simplicity, we drop index j_0 and write

$$u_n = \mathcal{G}_n \psi + r_n, \quad \tilde{u}_n(t) = V_n(t) + e^{-t\partial_x^3}r_n$$

in what follows. Further, we have $\lim_{n \rightarrow \infty} \|r_n\|_{\hat{M}_{2,\sigma}^\alpha} = 0$ and so

$$\lim_{n \rightarrow \infty} \|e^{-t\partial_x^3}r_n\|_{K \cap Z} = 0.$$

When $|\xi_n| \rightarrow \infty$, as in the previous step, we see from assumption (1.8) and Theorem 4.4 that $u_n \in S_+$ for large n , a contradiction. Hence, $\xi_n \equiv 0$. Recall that

$$V_n = D(h_n)T(y_n)\Psi((h_n)^3t + t_n)$$

where $\Psi(t)$ is a nonlinear profile associated with (ψ, t_n) . Let us now show that $u_c := \Psi$ is the solution which has the desired property. We have $\Psi(t_n) \notin S_+$, otherwise $u_n \in S_+$ for large n by long time stability.

The case $t_n \rightarrow \infty$ ($n \rightarrow \infty$) is excluded since this implies $\Psi(t_n) \in S_+$. If $t_n \equiv 0$ then $\Psi(0) = \psi$ and so $\ell(\Psi(0)) = d_+$. Finally, if $t_n \rightarrow -\infty$ as $n \rightarrow \infty$ then $\lim_{t \rightarrow -\infty} e^{t\partial_x^3}\Psi(t) = \psi$ and putting $u_{c,-} := \lim_{t \rightarrow -\infty} e^{t\partial_x^3}\Psi(t)$, we have $\ell(u_{c,-}) = d_+$. This completes the proof of Theorem 1.2. \square

4.3. Proof of Theorem 1.5. The proof is essentially the same as for Theorem 1.2. We first take a minimizing sequence associated with \tilde{d}_+ . Then, we apply Theorem 4.3. The difference is that uniform boundedness in $\hat{L}^{\tilde{\alpha}} \cap H^s$ gives us $h_n^j \equiv 1$ and $\xi_n^j \equiv 0$ (See Proposition 6.14). Thus, the assumption (1.8) is not necessary any longer since it is necessary just to exclude the case $\xi_n^j \rightarrow \infty$ via Theorem 4.4. Recall that $\tilde{B}_M \subset B_M$. Hence, the rest of the proof is the same. This completes the proof of Theorem 1.5.

5. LINEAR PROFILE DECOMPOSITION

In this section and the next section, we prove the linear profile decomposition (Theorem 4.3). To clarify the proof, we first prove a decomposition of sequence of *complex-valued* functions (Theorem 6.2, below). The desired decomposition for real-valued functions then follows as a corollary.

As in [8], the proof splits into two parts. The first part, treated in this section as Theorem 5.2, is the procedure of finding profiles and obtaining pairwise orthogonality between profiles and remainder term. By employing a successive notion of smallness of remainder term, used in [1, 8, 29], this part can be shown in an abstract way. Remark that, in fact, even a set of deformations need not to be specified for this decomposition procedure. For given boundedness and a set of deformations, a corresponding notion of orthogonality is selected and a corresponding decomposition is obtained. The set of the deformations

$$G := \{D(h)A(s)T(y)P(\xi) \mid \Gamma = (h, \xi, s, y) \in 2^{\mathbb{Z}} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}\}$$

used in Theorem 4.3 comes not from the decomposition procedure but from a concentration compactness, which is the second part of the proof and to be proven in the next section.

Remark 5.1. The above G plays a role of a *group of dislocations* in the sense of [49]. Remark that, however, G is not a group.

As explained in the introduction, a decoupling equality (1.3) fails by the presence of multiplier-like deformations A and T , and so the main point of our decomposition is to establish a decoupling *inequality* with respect to $\ell(\cdot)$ (Lemma 5.5).

Let us now state the main result of this section. For a bounded sequence $P = \{P_n\}_n \subset \hat{L}^\alpha$, we introduce a *set of weak limits modulo deformations*

$$\mathcal{V}(P) := \left\{ \phi \in \hat{L}^\alpha \left| \begin{array}{l} \phi = \lim_{k \rightarrow \infty} \mathcal{G}_{n_k}^{-1} P_{n_k} \text{ weakly in } \hat{L}^\alpha, \\ \exists \mathcal{G}_n \in G, \exists \text{subsequence } n_k \end{array} \right. \right\}.$$

and define

$$\eta(P) := \sup_{\phi \in \mathcal{V}(P)} \ell(\phi).$$

By definition, $\eta(P) = 0$ implies that we may not find any weak limit from a sequence $\{P_n\}_n$ even modulo the orbit by deformations G . Conversely, if $\eta(P) > 0$ we can find a non-zero weak limit modulo G . The main result of this section is decomposition with a smallness of remainder with respect to η .

Theorem 5.2. *Let $4/3 < \alpha < 2$ and $\alpha' < \sigma \leq \frac{6\alpha}{3\alpha-2}$. Let $u = \{u_n\}_n$ be a bounded sequence of \mathbb{C} -valued functions in \hat{L}^α . Then, there exist $\psi^j \in \mathcal{V}(u)$ and pairwise orthogonal families $\{\mathcal{G}_n^j\}_n \subset G$ ($j = 1, 2, \dots$) such that*

$$(5.1) \quad u_n = \sum_{j=1}^l \mathcal{G}_n^j \psi^j + r_n^l$$

for all $l \geq 1$ with

$$(5.2) \quad \eta(r^l) \rightarrow 0$$

as $l \rightarrow \infty$. Further, a decoupling inequality

$$(5.3) \quad \limsup_{n \rightarrow \infty} \ell(u_n)^\sigma \geq \sum_{j=1}^J \ell(\psi^j)^\sigma + \limsup_{n \rightarrow \infty} \ell(r_n^J)^\sigma$$

holds for all $J \geq 1$. Further, it holds that

$$(5.4) \quad \|\psi^j\|_{\hat{L}^\alpha} \leq \limsup_{n \rightarrow \infty} \|u_n\|_{\hat{L}^\alpha}$$

and

$$(5.5) \quad \limsup_{n \rightarrow \infty} \|r_n^j\|_{\hat{L}^\alpha} \leq (j+1) \limsup_{n \rightarrow \infty} \|u_n\|_{\hat{L}^\alpha}$$

for any j .

Remark 5.3. (i) The important thing in decoupling inequality (5.3) is that the coefficients of the right hand is equal to one. This is why we work not with $\|\cdot\|_{\hat{M}_{2,\sigma}^\alpha}$ but with $\ell(\cdot)$.

(ii) Unlikely to Theorem 4.3, the parameters ξ_n^j can be negative in this theorem. As a result, the notion of orthogonality is weaker. See Remark 6.7 for details.

5.1. A characterization of orthogonality. To begin with, we give a characterization of orthogonality of two families of deformations given in Definition 4.1. The orthogonality is realized as a condition which gives us the following two properties.

Lemma 5.4 (Characterization of orthogonality). *Let $\mathcal{G}_n, \tilde{\mathcal{G}}_n \in G$ be two families of deformations. The following three statements are equivalent.*

(i) \mathcal{G}_n and $\tilde{\mathcal{G}}_n$ are orthogonal.

(ii) It holds that

$$(\tilde{\mathcal{G}}_n)^{-1} \mathcal{G}_n \psi \rightharpoonup 0 \quad \text{weakly in } \hat{L}^\alpha$$

as $n \rightarrow \infty$ for any $\psi \in \hat{L}^\alpha$.

(iii) For any subsequence of n_k there exists a sequence $u_k \in \hat{L}^\alpha$ such that, up to subsequence (of k),

$$(\mathcal{G}_{n_k})^{-1} u_k \rightharpoonup \psi \neq 0, \quad (\tilde{\mathcal{G}}_{n_k})^{-1} u_k \rightharpoonup 0$$

weakly in \hat{L}^α as $k \rightarrow \infty$.

Proof. “(ii) \Rightarrow (iii)” is immediate by taking $u_k = (\mathcal{G}_{n_k})\psi$ for some $\psi \neq 0$.

We prove “(i) \Rightarrow (ii)”. Remark that the stated weak convergence is equivalent to

$$\mathcal{F}(\tilde{\mathcal{G}}_n)^{-1} \mathcal{G}_n \psi = (\tilde{\mathcal{G}}_n)^{-1} \hat{\mathcal{G}}_n \mathcal{F} \psi \rightharpoonup 0 \quad \text{weakly in } L^{\alpha'}$$

as $n \rightarrow \infty$. Set

$$\begin{aligned} h'_n &= \frac{h_n}{\tilde{h}_n}, & \xi'_n &= \xi_n - \frac{\tilde{h}_n}{h_n} \tilde{\xi}_n, \\ s'_n &= s_n - \left(\frac{h_n}{\tilde{h}_n} \right)^3 \tilde{s}_n, & y'_n &= y_n - \frac{h_n}{\tilde{h}_n} \tilde{y}_n - 3 \left(s_n - \left(\frac{h_n}{\tilde{h}_n} \right)^3 \tilde{s}_n \right) \xi_n^2 \end{aligned}$$

By density argument and (2.1), it suffices to show that

$$\int \mathbf{1}_K(\xi) [\hat{D}(h'_n) \hat{P}(\xi'_n) \hat{A}(s'_n) \hat{T}(y'_n) \hat{S}(3\xi_n s'_n) \mathbf{1}_L](\xi) d\xi \rightarrow 0$$

as $n \rightarrow \infty$ for any compact intervals K, L . When $|\log h'_n| \rightarrow \infty$ as $n \rightarrow \infty$, we have

$$\begin{aligned} & \left| \int \mathbf{1}_K(\xi) [\hat{D}(h'_n) \hat{P}(\xi'_n) \hat{A}(s'_n) \hat{T}(y'_n) \hat{S}(3\xi_n s'_n) \mathbf{1}_L](\xi) d\xi \right| \\ & \leq \min \left((h'_n)^{-1+\frac{1}{r}} |K|, (h'_n)^{\frac{1}{r}} |L| \right) \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. When $\limsup_{n \rightarrow \infty} |\log h'_n| < \infty$ and $|\xi'_n| \rightarrow \infty$ as $n \rightarrow \infty$, K and support of $\hat{D}(h'_n) \hat{P}(\xi'_n) \mathbf{1}_L$ are disjoint for large n and so we have the desired estimate.

We hence assume that $\limsup_{n \rightarrow \infty} (|\log h'_n| + |\xi'_n|) < \infty$. Taking subsequence, we may suppose that $h'_n \rightarrow h' \in (0, \infty)$ and $\xi'_n \rightarrow \xi' \in \mathbb{R}$ as $n \rightarrow \infty$. Then, for any $f \in \hat{L}^\alpha$, $D(h'_n) P(\xi'_n) f$ converges to $D(h') P(\xi') f$ strongly in \hat{L}^α . Since $D(h') P(\xi')$ is invertible, we need to show that $A(s'_n) T(y'_n) S(3\xi_n \tau'_n) \psi \rightharpoonup 0$ weakly in \hat{L}^α as $n \rightarrow \infty$. To do so, it suffices to show

$$\int [\hat{A}(s'_n) \hat{T}(y'_n) \hat{S}(3\xi_n s'_n) \mathbf{1}_L](\xi) d\xi \rightarrow 0$$

as $n \rightarrow \infty$ for any compact interval L . Put $\Phi_n(\xi) := s'_n \xi^3 - 3s'_n \xi_n \xi^2 - y'_n \xi$. Then, the right hand side is written as

$$\int_L e^{i\Phi_n(\xi)} d\xi.$$

We first consider the case $|\xi_n| \rightarrow \infty$. By orthogonality assumption, we have $|s'_n \xi_n| + |y'_n| \rightarrow \infty$ as $n \rightarrow \infty$. Hence, we see by an elementary computation that $\inf_L |\Phi'_n| \rightarrow \infty$ and $\sup_L |\Phi''_n|/|\Phi'_n|^2 \rightarrow 0$ as $n \rightarrow \infty$. Notice that there exists a special case such that $s'_n \rightarrow 0$, $|s'_n \xi_n| + |y'_n| \rightarrow \infty$, and $(6\eta s'_n \xi_n + y'_n) \rightarrow 0$ as $n \rightarrow \infty$ for some constant $\eta \in \mathbb{R}$. In such a case, we may assume that $\eta \notin L$ by density. Now, the desired smallness follows by integration by parts;

$$\begin{aligned} \left| \int_L e^{i\Phi_n(\xi)} d\xi \right| &\leq \left| \left[\frac{e^{i\Phi_n(\xi)}}{i\Phi'_n(\xi)} \right]_{\inf L}^{\sup L} + \int_L \frac{\Phi''_n(\xi)}{i(\Phi'_n(\xi))^2} e^{i\Phi_n(\xi)} d\xi \right| \\ &\leq L \left(\frac{1}{\inf_L |\Phi'_n|} + \sup_L \frac{|\Phi''_n|}{|\Phi'_n|^2} \right) \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. The proof for the case $\sup_n |\xi_n| < \infty$ is similar. In this case, we have $|s'_n| + |y'_n| \rightarrow \infty$ as $n \rightarrow \infty$ by orthogonality condition. Since

$$\Phi'_n(\xi) = s'_n(3\xi^2 - 6\xi_n \xi) - y'_n, \quad \Phi''_n(\xi) = 6s'_n(\xi - \xi_n),$$

by removing from L the constant η such that $\Phi'_n(\eta) \rightarrow 0$ as $n \rightarrow \infty$ if exists, we obtain the smallness.

Let us proceed to the proof of “(iii) \Rightarrow (i)”. Assume for contradiction that h'_n , ξ'_n , s'_n , $3s'_n \xi_n$, and y'_n are uniformly bounded. Then, there exists a subsequence n_k such that these parameters converge as $k \rightarrow \infty$. Denote the limits by h' , ξ' , s' , τ' , and y' , respectively. By refining subsequence if necessary, we may suppose that $e^{i\gamma_{n_k}}$ also converges. In this case, for any $f \in \hat{L}^\alpha$ we have

$$(5.6) \quad (\tilde{\mathcal{G}}_{n_k})^{-1} \mathcal{G}_{n_k} f \rightarrow e^{i\gamma} D(h') P(\xi') A(s') S(\tau') T(y') f$$

as $k \rightarrow \infty$ strongly in \hat{L}^α . Now, suppose that there exists a subsequence of k , which we denote again by k , such that $(\mathcal{G}_{n_k})^{-1} u_k$ and $(G_{n_k})^{-1} u_k$ converge weakly in \hat{L}^α to ψ and 0, respectively, as $k \rightarrow \infty$. Since $(\mathcal{G}_{n_k})^{-1} u_k$ converges to ψ weakly in \hat{L}^α , we see from (5.6) that

$$(\tilde{\mathcal{G}}_{n_k})^{-1} u_k = ((\tilde{\mathcal{G}}_{n_k})^{-1} \mathcal{G}_n)(\mathcal{G}_n)^{-1} u_n \rightharpoonup e^{i\gamma} D(h') P(\xi') A(s') S(\tau') T(y') \psi$$

weakly in \hat{L}^α . On the other hand, $(\tilde{\mathcal{G}}_{n_k})^{-1} u_k \rightharpoonup 0$ weakly in \hat{L}^α by assumption. Thanks to uniqueness of weak limit, we see that

$$e^{i\gamma} D(h') P(\xi') A(s') S(\tau') T(y') \psi = 0.$$

Thus, we obtain $\psi \equiv 0$, a contradiction. \square

5.2. Decoupling inequality. We next prove a decoupling inequality for ℓ . The idea of the proof is to sum up the local (in the Fourier side) L^2 decoupling with respect to intervals.

Lemma 5.5 (Decoupling inequality). *Let $4/3 < \alpha < 2$ and $\alpha' < \sigma \leq \frac{6\alpha}{3\alpha-2}$. Let $\{u_n\}_n$ be a bounded sequence in \hat{L}^α . Suppose that $\mathcal{G}_n^{-1} u_n$ converges to ψ*

weakly in \hat{L}^α as $n \rightarrow \infty$ with some $\{\mathcal{G}_n\}_n \subset G$. Set $r_n := u_n - \mathcal{G}_n\psi$. Then, for any $\gamma > 1$ and $\xi_0 \in \mathbb{R}$, it holds that

$$(5.7) \quad \gamma \|P(\xi_0)u_n\|_{\hat{M}_{2,\sigma}^\alpha}^\sigma \geq \|P(\xi_0)\mathcal{G}_n\psi\|_{\hat{M}_{2,\sigma}^\alpha}^\sigma + \|P(\xi_0)r_n\|_{\hat{M}_{2,\sigma}^\alpha}^\sigma + o_\gamma(1)$$

as $n \rightarrow \infty$.

Proof. We only consider the case $\xi_0 = 0$. The other cases handled in the same way because the presence of $P(\xi_0)$ causes merely a universal translation in the Fourier side. It is also clear from the proof that the small error term can be taken independently of ξ_0 .

Denote $\mathcal{D} := \{\tau_k^l := [k/2^l, (k+1)/2^l) \mid k, l \in \mathbb{Z}\}$. For each $\tau_k^l \in \mathcal{D}$, we have the decoupling in L^2 ;

$$\|\mathcal{F}u_n\|_{L^2(\tau_k^l)}^2 = \|\mathcal{F}\mathcal{G}_n\psi\|_{L^2(\tau_k^l)}^2 + \|\mathcal{F}r_n\|_{L^2(\tau_k^l)}^2 + 2 \operatorname{Re} \langle \mathcal{F}\mathcal{G}_n\psi, \mathcal{F}r_n \rangle_{\tau_k^l}.$$

Let $\gamma = \frac{m+1}{m}$, $m > 0$. By an elementary inequality

$$(a - b)^{\frac{\sigma}{2}} \geq \left(\frac{m}{m+1} \right)^{\frac{\sigma-2}{2}} a^{\frac{\sigma}{2}} - m^{\frac{\sigma-2}{2}} b^{\frac{\sigma}{2}}$$

for any $a \geq b \geq 0$ and $m > 0$ and by embedding $\ell_{\mathcal{D}}^2 \hookrightarrow \ell_{\mathcal{D}}^\sigma$, it follows that

$$\begin{aligned} & \sum_{k,l \in \mathbb{Z}} |\tau_k^l|^{\sigma(\frac{1}{2}-\frac{1}{\alpha})} \|\mathcal{F}u_n\|_{L^2(\tau_k^l)}^\sigma \\ &= \sum_{k,l \in \mathbb{Z}} |\tau_k^l|^{\sigma(\frac{1}{2}-\frac{1}{\alpha})} \left(\|\mathcal{F}\mathcal{G}_n\psi\|_{L^2(\tau_k^l)}^2 + \|\mathcal{F}r_n\|_{L^2(\tau_k^l)}^2 + 2 \operatorname{Re} \langle \mathcal{F}\mathcal{G}_n\psi, \mathcal{F}r_n \rangle_{\tau_k^l} \right)^{\frac{\sigma}{2}} \\ &\geq \sum_{k,l \in \mathbb{Z}} |\tau_k^l|^{\sigma(\frac{1}{2}-\frac{1}{\alpha})} \left(\|\mathcal{F}\mathcal{G}_n\psi\|_{L^2(\tau_k^l)}^2 + \|\mathcal{F}r_n\|_{L^2(\tau_k^l)}^2 - 2 \left| \langle \mathcal{F}\mathcal{G}_n\psi, \mathcal{F}r_n \rangle_{\tau_k^l} \right| \right)^{\frac{\sigma}{2}} \\ &\geq \left(\frac{m}{m+1} \right)^{\frac{\sigma-2}{2}} \sum_{k,l \in \mathbb{Z}} |\tau_k^l|^{\sigma(\frac{1}{2}-\frac{1}{\alpha})} \left(\|\mathcal{F}\mathcal{G}_n\psi\|_{L^2(\tau_k^l)}^2 + \|\mathcal{F}r_n\|_{L^2(\tau_k^l)}^2 \right)^{\frac{\sigma}{2}} \\ &\quad - 2^{\frac{\sigma}{2}} m^{\frac{\sigma-2}{2}} \sum_{k,l \in \mathbb{Z}} |\tau_k^l|^{\sigma(\frac{1}{2}-\frac{1}{\alpha})} \left| \langle \mathcal{F}\mathcal{G}_n\psi, \mathcal{F}r_n \rangle_{\tau_k^l} \right|^{\frac{\sigma}{2}} \\ &\geq \left(\frac{m}{m+1} \right)^{\frac{\sigma-2}{2}} \left(\sum_{k,l \in \mathbb{Z}} |\tau_k^l|^{\sigma(\frac{1}{2}-\frac{1}{\alpha})} \|\mathcal{F}\mathcal{G}_n\psi\|_{L^2(\tau_k^l)}^\sigma + \sum_{k,l \in \mathbb{Z}} |\tau_k^l|^{\sigma(\frac{1}{2}-\frac{1}{\alpha})} \|\mathcal{F}r_n\|_{L^2(\tau_k^l)}^\sigma \right) \\ &\quad - 2^{\frac{\sigma}{2}} m^{\frac{\sigma-2}{2}} \sum_{k,l \in \mathbb{Z}} |\tau_k^l|^{\sigma(\frac{1}{2}-\frac{1}{\alpha})} \left| \langle \mathcal{F}\mathcal{G}_n\psi, \mathcal{F}r_n \rangle_{\tau_k^l} \right|^{\frac{\sigma}{2}} \end{aligned}$$

To obtain (5.7), it therefore suffices to show that

$$(5.8) \quad R_n := \sum_{k,l \in \mathbb{Z}} |\tau_k^l|^{\sigma(\frac{1}{2}-\frac{1}{\alpha})} \left| \langle \mathcal{F}\mathcal{G}_n\psi, \mathcal{F}r_n \rangle_{\tau_k^l} \right|^{\frac{\sigma}{2}} \rightarrow 0$$

as $n \rightarrow \infty$. A computation shows that

$$|I|^{\sigma(\frac{1}{2}-\frac{1}{\alpha})} |\langle \mathcal{F}\mathcal{G}_n\psi, \mathcal{F}r_n \rangle_I|^{\frac{\sigma}{2}} = |J_n|^{\sigma(\frac{1}{2}-\frac{1}{\alpha})} \left| \langle \mathcal{F}\psi, \mathcal{F}(\mathcal{G}_n)^{-1}r_n \rangle_{J_n} \right|^{\frac{\sigma}{2}}$$

for any $I \subset \mathbb{R}$, where $J_n = I/h_n + \xi_n$ with the parameters h_n, ξ_n associated with \mathcal{G}_n . By changing notation if necessary, one sees that

$$R_n = \sum_{k,l \in \mathbb{Z}} |\tilde{\tau}_k^l|^{\sigma(\frac{1}{2}-\frac{1}{\alpha})} \left| \langle \mathcal{F}\psi^1, \mathcal{F}(\mathcal{G}_n^1)^{-1} r_n^1 \rangle_{\tilde{\tau}_k^l} \right|^{\frac{\sigma}{2}},$$

where $\tilde{\tau}_k^l = \tau_k^l + 2^{-l}\sigma_n$ with some $0 \leq \sigma_n < 1$. Fix $\varepsilon > 0$. Since $\|\psi\|_{\dot{M}_{2,\sigma}^\alpha} \leq C \|\psi\|_{\dot{L}^\alpha} < \infty$, there exist $k_0(\varepsilon)$ and $l_0(\varepsilon)$ such that $D := \{|l| \leq l_0, |k| \leq k_0\} \subset \mathbb{Z}^2$ satisfies

$$\sum_{(k,l) \in \mathbb{Z}^2 \setminus D} |\tau_k^l|^{\sigma(\frac{1}{2}-\frac{1}{\alpha})} \|\mathcal{F}\psi\|_{L^2(\tau_k^l)}^\sigma \leq \varepsilon.$$

It is obvious that $\tilde{\tau}_k^l \subset \tau_k^l \cup \tau_{k+1}^l$ and $|\tilde{\tau}_k^l| = |\tau_k^l| = |\tau_{k+1}^l|$ for each l, k . Hence, denoting $D' := \{|l| \leq l_0, |k| \leq k_0 + 1\} \subset \mathbb{Z}^2$, we have

$$\begin{aligned} & \sum_{(k,l) \in \mathbb{Z}^2 \setminus D'} |\tilde{\tau}_k^l|^{\sigma(\frac{1}{2}-\frac{1}{\alpha})} \|\mathcal{F}\psi^1\|_{L^2(\tilde{\tau}_k^l)}^\sigma \\ & \leq \sum_{(k,l) \in \mathbb{Z}^2 \setminus D'} |\tilde{\tau}_k^l|^{\sigma(\frac{1}{2}-\frac{1}{\alpha})} \left(\|\mathcal{F}\psi^1\|_{L^2(\tau_k^l)}^2 + \|\mathcal{F}\psi^1\|_{L^2(\tau_{k+1}^l)}^2 \right)^{\sigma/2} \\ & \leq 2^{\frac{\sigma}{2}} \sum_{(k,l) \in \mathbb{Z}^2 \setminus D} |\tau_k^l|^{\sigma(\frac{1}{2}-\frac{1}{\alpha})} \|\mathcal{F}\psi^1\|_{L^2(\tau_k^l)}^\sigma \leq C\varepsilon. \end{aligned}$$

Then, by Schwartz' inequality,

$$\begin{aligned} & \sum_{(k,l) \in \mathbb{Z}^2 \setminus D'} |\tilde{\tau}_k^l|^{\sigma(\frac{1}{2}-\frac{1}{\alpha})} \left| \langle \mathcal{F}\psi^1, \mathcal{F}(\mathcal{G}_n)^{-1} r_n^1 \rangle_{\tilde{\tau}_k^l} \right|^{\frac{\sigma}{2}} \\ & \leq \sum_{(k,l) \in \mathbb{Z}^2 \setminus D'} |\tilde{\tau}_k^l|^{\sigma(\frac{1}{2}-\frac{1}{\alpha})} \|\mathcal{F}\psi^1\|_{L^2(\tilde{\tau}_k^l)}^{\frac{\sigma}{2}} \|\mathcal{F}(\mathcal{G}_n)^{-1} r_n^1\|_{L^2(\tilde{\tau}_k^l)}^{\frac{\sigma}{2}} \\ & \leq \left(\sum_{(k,l) \in \mathbb{Z}^2 \setminus D'} |\tilde{\tau}_k^l|^{\sigma(\frac{1}{2}-\frac{1}{\alpha})} \|\mathcal{F}\psi^1\|_{L^2(\tilde{\tau}_k^l)}^\sigma \right)^{\frac{1}{2}} \\ & \quad \times \left(\sum_{(k,l) \in \mathbb{Z}^2 \setminus D'} |\tilde{\tau}_k^l|^{\sigma(\frac{1}{2}-\frac{1}{\alpha})} \|\mathcal{F}(\mathcal{G}_n)^{-1} r_n^1\|_{L^2(\tilde{\tau}_k^l)}^\sigma \right)^{\frac{1}{2}} \\ & \leq C\varepsilon^{\frac{1}{2}} \|(\mathcal{G}_n)^{-1} r_n^1\|_{\dot{M}_{2,\sigma}^\alpha}^{\frac{\sigma}{2}} \leq C\varepsilon^{\frac{1}{2}} \|(\mathcal{G}_n)^{-1} r_n^1\|_{\dot{L}^\alpha}^{\frac{\sigma}{2}} \\ & = C\varepsilon^{\frac{1}{2}} \|r_n^1\|_{\dot{L}^\alpha}^{\frac{\sigma}{2}}. \end{aligned}$$

Remark that

$$\limsup_{n \rightarrow \infty} \|r_n\|_{\dot{L}^\alpha} \leq \limsup_{n \rightarrow \infty} \|u_n\|_{\dot{L}^\alpha} + \|\psi\|_{\dot{L}^\alpha} \leq 2 \limsup_{n \rightarrow \infty} \|u_n\|_{\dot{L}^\alpha} \leq C.$$

Hence, the proof of (5.8) is reduced to showing

$$(5.9) \quad \sum_{(k,l) \in D'} |\tilde{\tau}_k^l|^{\sigma(\frac{1}{2}-\frac{1}{\alpha})} \left| \langle \mathcal{F}\psi^1, \mathcal{F}(\mathcal{G}_n)^{-1} r_n^1 \rangle_{\tilde{\tau}_k^l} \right|^{\frac{\sigma}{2}} \rightarrow 0$$

as $n \rightarrow \infty$. For $l \in [-l_0, l_0]$, consider a function

$$f_n^l(x) := \left| \langle \mathcal{F}\psi^1, \mathcal{F}(\mathcal{G}_n)^{-1}r_n^1 \rangle_{[x, x+2^{-l}]} \right|$$

with domain $x \in [-k_0/2^l, (k_0+1)/2^l]$. Then, there exists a constant $C = C(k_0, l_0) = C(\varepsilon)$ such that

$$\begin{aligned} & \sum_{(k,l) \in D'} |\tilde{\tau}_k^l|^{\sigma(\frac{1}{2}-\frac{1}{\alpha})} \left| \langle \mathcal{F}\psi^1, \mathcal{F}(\mathcal{G}_n)^{-1}r_n^1 \rangle_{\tilde{\tau}_k^l} \right|^{\frac{\sigma}{2}} \\ & \leq C(\varepsilon) \max_{l \in [-l_0, l_0]} \left(\sup_{x \in [-k_0/2^l, (k_0+1)/2^l]} f_n^l(x) \right). \end{aligned}$$

Therefore, we obtain (5.9) if we show the uniform convergence

$$(5.10) \quad \sup_{x \in [-k_0/2^l, (k_0+1)/2^l]} f_n^l(x) \rightarrow 0$$

as $n \rightarrow \infty$. Since $(\mathcal{G}_n)^{-1}r_n$ converges to zero weakly in \hat{L}^α as $n \rightarrow \infty$ by definition, $\lim_{n \rightarrow \infty} f_n(x) = 0$ follows for each x . Further, by the Hölder inequality,

$$\begin{aligned} & |f_n^l(x+\delta) - f_n^l(x)| \\ & \leq C \left(\sup_n \|(\mathcal{G}_n)^{-1}r_n^1\|_{\hat{L}^\alpha} \right) \|\mathcal{F}\psi^1\|_{L^\alpha([x, x+\delta] \cup [x+2^{-l}, x+2^{-l}+\delta])} \end{aligned}$$

for small $\delta > 0$. The right hand side is independent of n and tends to zero as $\delta \downarrow 0$. Therefore, $\{f_n^l\}_n$ is equicontinuous. By a similar argument, $\sup_{x \in [-k_0/2^l, (k_0+1)/2^l]} f_n^l(x)$ is bounded uniformly in n . Therefore, the Ascoli-Arzelà theorem gives us the desired convergence (5.10). This completes the proof of Lemma 5.5. \square

5.3. Decomposition procedure. The main technical issue of Theorem 5.2 is essentially settled with the above preliminaries and so now the theorem follows by a standard argument (see [8] and references therein). We give a complete proof for self-containedness and in order to give a complete proof for the decoupling inequality (5.3).

Proof of Theorem 5.2. We may suppose $\eta(u) > 0$, otherwise the result holds with $\phi^j \equiv 0$ and $r_n^j = u_n$ for all $j \geq 1$. Then, we can choose $\psi^1 \in \mathcal{V}(u)$ so that $\ell(\psi^1) \geq \frac{1}{2}\eta(u)$ by definition of η . Then, by definition of $\mathcal{V}(u)$, one finds $\mathcal{G}_n^1 \in G$ such that

$$(\mathcal{G}_n^1)^{-1}u_n \rightharpoonup \psi^1 \quad \text{weakly in } \hat{L}^\alpha$$

as $n \rightarrow \infty$ up to subsequence. By lower semicontinuity of weak limit, we obtain (5.4) for $j = 1$. Define $r_n^1 := u_n - \mathcal{G}_n^1\psi^1$. Then, it is obvious that

$$(5.11) \quad (\mathcal{G}_n^1)^{-1}r_n^1 \rightharpoonup \psi^1 - \psi^1 = 0 \quad \text{weakly in } \hat{L}^\alpha$$

as $n \rightarrow \infty$. The boundedness (5.5) for $j = 1$ is also obvious by (5.4). By Lemma 5.5,

$$(5.12) \quad \gamma \|P(\xi_0)u_n\|_{\dot{M}_{2,\sigma}^\alpha}^\sigma \geq \|P(\xi_0)\mathcal{G}_n^1\psi^1\|_{\dot{M}_{2,\sigma}^\alpha}^\sigma + \|P(\xi_0)r_n^1\|_{\dot{M}_{2,\sigma}^\alpha}^\sigma + o_\gamma(1)$$

as $n \rightarrow \infty$ for any constant $\gamma > 1$ and $\xi_0 \in \mathbb{R}$. Since $\gamma > 1$ and ξ_0 are arbitrary, the decoupling inequality (5.3) holds for $J = 1$.

If $\eta(r^1) = 0$ then the proof is completed by taking $\psi^j \equiv 0$ for $j \geq 2$. Otherwise, we can choose $\psi^2 \in \mathcal{V}(r^1)$ so that $\ell(\psi^2) \geq \frac{1}{2}\eta(r^1)$. Then, as in the previous step, one can take $\mathcal{G}_n^2 \in G$ so that

$$(\mathcal{G}_n^2)^{-1}r_n^1 \rightharpoonup \psi^2 \quad \text{weakly in } \hat{L}^\alpha$$

as $n \rightarrow \infty$ up to subsequence. In particular, $\psi^2 \neq 0$. Together with (5.11), Lemma 5.4 gives us that two families \mathcal{G}_n^1 and \mathcal{G}_n^2 are orthogonal. Then,

$$(\mathcal{G}_n^2)^{-1}u_n = (\mathcal{G}_n^2)^{-1}\mathcal{G}_n^1\psi^1 + (\mathcal{G}_n^2)^{-1}r_n^1 \rightharpoonup 0 + \psi^2 \quad \text{weakly in } \hat{L}^\alpha$$

as $n \rightarrow \infty$. Hence, we obtain $\psi^2 \in \mathcal{V}(u)$ and so (5.4) for $j = 2$. Set $r_n^2 := r_n^1 - \mathcal{G}_n^2\psi^2$. Then, (5.5) for $j = 2$ follows from

$$\limsup_{n \rightarrow \infty} \|r_n^2\|_{\hat{L}^\alpha} \leq \limsup_{n \rightarrow \infty} \|r_n^1\|_{\hat{L}^\alpha} + \|\psi^2\|_{\hat{L}^\alpha} \leq 3 \limsup_{n \rightarrow \infty} \|u_n\|_{\hat{L}^\alpha}.$$

Further, one deduces from Lemma 5.5 that

$$\gamma \|P(\xi_0)r_n^1\|_{\dot{M}_{2,\sigma}^\alpha}^\sigma \geq \|P(\xi_0)\mathcal{G}_n^2\psi^2\|_{\dot{M}_{2,\sigma}^\alpha}^\sigma + \|P(\xi_0)r_n^2\|_{\dot{M}_{2,\sigma}^\alpha}^\sigma + o_\gamma(1)$$

as $n \rightarrow \infty$ for any $\gamma > 1$ and $\xi_0 \in \mathbb{R}$. This implies (5.3) for $J = 2$ with the help of (5.12).

Repeat this argument and construct $\psi^j \in \mathcal{V}(r^{j-1})$ and $\mathcal{G}_n^j \in G$, inductively. If we have $\eta(r^{j_0}) = 0$ for some j_0 , then we define $\psi^j \equiv 0$ for $j \geq j_0 + 1$. In what follows, we may suppose that $\eta(r^j) > 0$ for all $j \geq 1$. In each step, r_n^j is defined by the formula $r_n^j = r_n^{j-1} - \mathcal{G}_n^j\psi^j$. The property (5.1) is obvious by construction.

Let us now prove that pairwise orthogonality. To this end, we demonstrate that \mathcal{G}_n^j is orthogonal to \mathcal{G}_n^k for $1 \leq k \leq j-1$. Since $(\mathcal{G}_n^j)^{-1}r_n^j \rightharpoonup \psi^j$ and $(\mathcal{G}_n^{j-1})^{-1}r_n^j \rightharpoonup 0$ in \hat{L}^α as $n \rightarrow \infty$, Lemma 5.4 implies that \mathcal{G}_n^j and \mathcal{G}_n^{j-1} are orthogonal. If \mathcal{G}_n^j is orthogonal to \mathcal{G}_n^k for $k_0 \leq k \leq j-1$ then Lemma 5.4 yields

$$(\mathcal{G}_n^j)^{-1}r_n^{k_0-1} = \sum_{k=k_0}^{j-1} (\mathcal{G}_n^j)^{-1}\mathcal{G}_n^k\psi^k + (\mathcal{G}_n^j)^{-1}r_n^{j-1} \rightharpoonup \psi^j$$

as $n \rightarrow \infty$. On the other hand, $(\mathcal{G}_n^{k_0-1})^{-1}r_n^{k_0-1} \rightharpoonup 0$ as $n \rightarrow \infty$. We therefore see from Lemma 5.4 that \mathcal{G}_n^j and $\mathcal{G}_n^{k_0-1}$ are orthogonal. Hence, \mathcal{G}_n^j is orthogonal to \mathcal{G}_n^k for $1 \leq k \leq j-1$. Then, by (5.1) and by Lemma 5.4, we have $\psi^j \in \mathcal{V}(u)$, from which boundedness (5.4) and (5.5) follow.

To conclude the proof, we shall show (5.2) and (5.3). Notice that the inductive construction gives us

$$(5.13) \quad \ell(\psi^{j+1}) \geq \frac{1}{2}\eta(r^j)$$

for $j \geq 1$ and

$$(5.14) \quad \begin{aligned} & \gamma \|P(\xi_0)r_n^j\|_{\dot{M}_{2,\sigma}^\alpha}^\sigma \\ & \geq \|P(\xi_0)\mathcal{G}_n^{j+1}\psi^{j+1}\|_{\dot{M}_{2,\sigma}^\alpha}^\sigma + \|P(\xi_0)r_n^{j+1}\|_{\dot{M}_{2,\sigma}^\alpha}^\sigma + o_{\gamma,j}(1). \end{aligned}$$

as $n \rightarrow \infty$ for (fixed) $j \geq 1$ and any $\gamma > 1$ and $\xi_0 \in \mathbb{R}$. Combining (5.12) and (5.14) for $1 \leq j \leq J$, we have

$$\begin{aligned} \gamma^J \|P(\xi_0)u_n\|_{\hat{M}_{2,\sigma}^\alpha}^\sigma &\geq \sum_{j=1}^J \gamma^{J-j} \|P(\xi_0)\mathcal{G}_n^j \psi^j\|_{\hat{M}_{2,\sigma}^\alpha}^\sigma + \|P(\xi_0)r_n^J\|_{\hat{M}_{2,\sigma}^\alpha}^\sigma + o_{\gamma,J}(1) \\ &\geq \sum_{j=1}^J \gamma^{J-j} \ell(\psi^j)^\sigma + \ell(r_n^J)^\sigma + o_{\gamma,J}(1). \end{aligned}$$

Take first infimum with respect to ξ_0 and then limit supremum in n to obtain

$$\limsup_{n \rightarrow \infty} \ell(u_n)^\sigma \geq \sum_{j=1}^J \gamma^{-j} \ell(\psi^j)^\sigma + \gamma^{-J} \limsup_{n \rightarrow \infty} \ell(r_n^J)^\sigma.$$

Since $\gamma > 1$ is arbitrary, we obtain (5.3). Finally, (5.3) and (5.13) imply (5.2). This completes the proof of Theorem 5.2.

6. CONCENTRATION COMPACTNESS

The second part of the proof of Theorem 4.3 is concentration compactness. Intuitively, the meaning of the concentration compactness is as follows. Let us consider a bonded sequence $\{u_n\}_n \subset X$. Here, X is a Banach space. In addition to the boundedness with respect to X , we make some additional assumption on the sequence. If the additional assumption is so strong that it removes almost all possible deformations for $\{u_n\}_n$ with few exceptions, say G , then we can find a *non-zero* weak limit modulo G . In our case, $X = \hat{M}_{2,\sigma}^\alpha$ and we use

$$\left\| e^{-t\partial_x^3} u_n \right\|_{L(\mathbb{R})} \geq m$$

as an additional assumption, where m is some positive constant. It will turn out that this assumption removes almost all deformations. The exception is G given in (4.1). This is the reason why we use the set G of deformations in Theorems 4.3 or 5.2. The precise statement is as follows.

Theorem 6.1 (Concentration compactness). *Let $4/3 < \alpha < 2$ and $\alpha' < \sigma < \frac{6\alpha}{3\alpha-2}$. Let a bounded sequence $\{u_n\} \subset \hat{L}^\alpha$ satisfy*

$$(6.1) \quad \|u_n\|_{\hat{M}_{2,\sigma}^\alpha} \leq M$$

and

$$(6.2) \quad \left\| e^{-t\partial_x^3} u_n \right\|_{L(\mathbb{R})} \geq m$$

for some positive constants m, M . Then, there exist $\mathcal{G}_n \in G$ and $\psi \in \hat{L}^\alpha$ such that, up to subsequence, $\mathcal{G}_n^{-1} u_n \rightharpoonup \psi$ weakly in \hat{L}^α as $n \rightarrow \infty$ and $\|u_n\|_{\hat{M}_{2,\sigma}^\alpha} \geq \beta(m, M)$, where $\beta(m, M)$ is a positive constant depending only on m, M . In particular, $\eta(u) \geq C\beta(m, M)$ holds for some constant C .

Plugging Theorem 6.1 to Theorem 5.2, we obtain desired decomposition result.

Theorem 6.2 (Decomposition of “complex-valued” functions). *Let $4/3 < \alpha < 2$ and $\sigma \in (\alpha', \frac{6\alpha}{3\alpha-2})$. Let $u = \{u_n\}_n$ be a bounded sequence of \mathbb{C} -valued functions in \hat{L}^α . Then, there exist $\{\psi^j\}_j, \{r_n^j\}_{n,j} \subset \hat{L}^\alpha$ and pairwise orthogonal families $\{\mathcal{G}_n^j\}_n \subset G$ ($j = 1, 2, \dots$) such that, up to subsequence,*

$$u_n = \sum_{j=1}^l \mathcal{G}_n^j \psi^j + r_n^l$$

for all $l \geq 1$ with

$$(6.3) \quad \limsup_{n \rightarrow \infty} \left\| e^{-t\partial_x^3} r_n^l \right\|_{L(\mathbb{R})} \rightarrow 0$$

as $l \rightarrow \infty$. Further, the decouple inequality

$$\limsup_{n \rightarrow \infty} \ell(u_n)^\sigma \geq \sum_{j=1}^{\infty} \ell(\psi^j)^\sigma + \limsup_{n \rightarrow \infty} \ell(r_n^J)^\sigma$$

holds for any $J \geq 1$. Moreover, it holds that

$$\|\psi^j\|_{\hat{L}^\alpha} \leq \limsup_{n \rightarrow \infty} \|u_n\|_{\hat{L}^\alpha}$$

for any j .

Before proceeding to the proof of Theorem 6.1, we demonstrate how we derive Theorem 6.2 from Theorems 5.2 and 6.1.

Proof. By means of Theorem 5.2, it suffices to show (6.3) as $l \rightarrow \infty$. Assume for contradiction that a sequence r_n^l given in Theorem 5.2 satisfies

$$\limsup_{l \rightarrow \infty} \limsup_{n \rightarrow \infty} \left\| e^{-t\partial_x^3} r_n^l \right\|_{L(\mathbb{R})} > 0.$$

Then, we can choose $m > 0$ and a subsequence l_k with $l_k \rightarrow \infty$ as $k \rightarrow \infty$ such that the assumption of Theorem 6.1 is fulfilled for each k . Then, Theorem 6.1 implies $\eta(r^{l_k}) \geq C\beta > 0$, which contradicts to (5.2). \square

Remark 6.3. We would emphasize that, in Theorem 6.1, $\{u_n\}_n$ should be a bounded sequence of \hat{L}^α functions but the constant β is chosen independently of the value of $\limsup_{n \rightarrow \infty} \|u_n\|_{\hat{L}^\alpha}$. This respect is crucial to obtain Theorem 6.2 because an \hat{L}^α -bound on r_n^J given in Theorem 5.2 is no more than (5.5).

By a similar argument, we obtain Theorem 1.8. We recall the theorem in terms of η .

Theorem 6.4 (Scattering due to irrelevant deformations). *Let $\{u_{0,n}\}_n \subset \hat{L}^\alpha$ be a bounded sequence. Let $u_n(t)$ be a solution to (gKdV) with $u_n(0) = u_{0,n}$. If $\eta(\{u_{0,n}\}_n) = 0$ then there exists N_0 such that $u_n(t)$ is global and scatters for both time direction as long as $n \geq N_0$. Furthermore,*

$$\|u_n\|_{S(\mathbb{R})} + \|u_n\|_{L(\mathbb{R})} \leq 2 \limsup_{n \rightarrow \infty} \|u_{0,n}\|_{\hat{L}^\alpha}$$

for $n \geq N_0$.

Proof. Just as in the proof of Theorem 6.2, we deduce from $\eta(\{u_{0,n}\}_n) = 0$ that

$$\lim_{n \rightarrow \infty} \left\| |\partial_x|^{\frac{1}{3\alpha}} e^{-t\partial_x^3} u_{0,n} \right\|_{L_{t,x}^{3\alpha}(\mathbb{R} \times \mathbb{R})} = 0$$

thanks to Theorem 6.1. Then, the result follows from Corollary 3.3. \square

6.1. Refined Stein-Tomas estimate. One of the key for the proof Theorem 6.1 is the following refinement of (2.4).

Theorem 6.5. *Let $4/3 < \alpha < 2$ and let σ satisfy $\sigma \in (0, \frac{6\alpha}{3\alpha-2})$. Then, there exists a positive constant $C = C(\alpha)$ such that*

$$\begin{aligned} \left\| e^{-t\partial_x^3} f \right\|_{L(\mathbb{R})} &\leq C \|f\|_{\dot{M}_{\frac{3\alpha}{2}, \frac{6\alpha}{3\alpha-2}}^\alpha} \\ &\leq C \|f\|_{\dot{M}_{\frac{3\alpha}{2}, \infty}^\alpha}^{1-\sigma(\frac{1}{2}-\frac{1}{3\alpha})} \|f\|_{\dot{M}_{\frac{3\alpha}{2}, \sigma}^\alpha}^{\sigma(\frac{1}{2}-\frac{1}{3\alpha})} \\ &\leq C \|f\|_{\dot{M}_{\frac{3\alpha}{2}, \infty}^\alpha}^{1-\sigma(\frac{1}{2}-\frac{1}{3\alpha})} \|f\|_{\dot{M}_{2,\sigma}^\alpha}^{\sigma(\frac{1}{2}-\frac{1}{3\alpha})} \\ &\leq C \|f\|_{\dot{M}_{2,\sigma}^\alpha} \end{aligned} \tag{6.4}$$

for any $f \in \dot{M}_{2,\sigma}^\alpha$.

This kind of refinement of the Airy Strichartz's inequality, is known in the case $\alpha = 2$ (see [28, 51]). We prove the first inequality of (6.4) in Appendix B. The others follow from embeddings in Remark 2.2 in Section 2.

6.2. A stronger orthogonality. We have settled the notion of orthogonality of two families of deformations in Definition 4.1. As seen in Lemma 5.4, the orthogonality is rather a property associated with the functions of space \hat{L}^α . The notion is not sufficient to yield weakness of interaction between Airy evolutions of them. A simple but essential example is

$$u_n = P(n)f_+ + P(-n)f_-,$$

where $f_\pm \in \mathcal{S}$ is a real-valued function good enough. Remark that $\{P(n)\}_n$ and $\{P(-n)\}_n$ are orthogonal. As for the Schrödinger evolution of u_n , we have the following decoupling with respect to a space-time norm

$$\|S(t)u_n\|_{L_{t,x}^p(\mathbb{R}^2)}^p = \|S(t)f_+\|_{L_{t,x}^p(\mathbb{R}^2)}^p + \|S(t)f_-\|_{L_{t,x}^p(\mathbb{R}^2)}^p + o(1).$$

for $4 < p < \infty$ as long as $\|S(t)f_\pm\|_{L_{t,x}^p(\mathbb{R}^2)}$ are finite. Indeed, the decoupling follows from

$$|S(\cdot)P(\pm n)f_\pm|(t, x) = |S(\cdot)f|(t, x \pm 2tn),$$

which is a consequence of the Galilean transform (1.7). In contrast, the Airy evolution of u_n may not satisfy such kind of space-time decoupling because

$$|A(\cdot)P(-n)f_-|(t, x) = |\overline{A(\cdot)P(n)f_-}|(t, x) = |A(\cdot)P(n)f_-|(t, x).$$

Thus, interaction between $A(\cdot)P(-n)f_-$ and $|A(\cdot)P(-n)f_+|$ is not always small.

Hence, we introduce a stronger notion of orthogonality which yields such a decoupling with respect to a space-time norm of corresponding Airy evolutions.

Definition 6.6. Let $\{\mathcal{G}_n\}_n, \{\tilde{\mathcal{G}}_n\}_n \subset G$ be two families of deformations and let $\Gamma_n, \tilde{\Gamma}_n \in 2^{\mathbb{Z}} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ be parameters associated with \mathcal{G}_n and $\tilde{\mathcal{G}}_n$, respectively. We say $\{\mathcal{G}_n\}_n$ and $\{\tilde{\mathcal{G}}_n\}_n$ are space-time nonresonant if

$$(6.5) \quad \limsup_{n \rightarrow \infty} \left(\left| \log \frac{h_n}{\tilde{h}_n} \right| + \left| |\xi_n| - \frac{\tilde{h}_n}{h_n} |\tilde{\xi}_n| \right| + \left| s_n - \left(\frac{h_n}{\tilde{h}_n} \right)^3 \tilde{s}_n \right| (1 + |\xi_n|) \right. \\ \left. + \left| y_n - \frac{h_n}{\tilde{h}_n} \tilde{y}_n - 3 \left(s_n - \left(\frac{h_n}{\tilde{h}_n} \right)^3 \tilde{s}_n \right) (\xi_n)^2 \right| \right) = +\infty.$$

Remark 6.7. (i) Obviously, if $\xi_n^j, \xi_n^k \geq 0$ then the orthogonality and the space-time nonresonant property of \mathcal{G}_n^j and \mathcal{G}_n^k are equivalent. Hence, we can replace the word “orthogonal” with “space-time nonresonant” in Theorem 4.3.

(ii) Let $\{\mathcal{G}_n^j\}_n \subset G$ ($j = 1, 2, \dots$) be pairwise orthogonal families. For each $\{\mathcal{G}_n^j\}_n$, the family $\{\mathcal{G}_n^k\}_n$ that is orthogonal but not space-time nonresonant is at most one. Indeed, if $\{\mathcal{G}_n^j\}_n$ and $\{\mathcal{G}_n^k\}_n$ are orthogonal and not space-time nonresonant, then

$$\limsup_{n \rightarrow \infty} \left(\left| \log \frac{h_n^j}{h_n^k} \right| + \left| \xi_n^j + \frac{h_n^k}{h_n^j} \xi_n^k \right| + \left| s_n^j - \left(\frac{h_n^j}{h_n^k} \right)^3 s_n^k \right| (1 + |\xi_n^j|) \right. \\ \left. + \left| y_n^j - \frac{h_n^j}{h_n^k} y_n^k - 3 \left(s_n^j - \left(\frac{h_n^j}{h_n^k} \right)^3 s_n^k \right) (\xi_n^j)^2 \right| \right) < +\infty$$

and

$$\limsup_{n \rightarrow \infty} \left| \xi_n^j - \frac{h_n^k}{h_n^j} \xi_n^k \right| = +\infty.$$

To simplify the formulation, we may assume that $h_n^j = h_n^k$. If both $\{\mathcal{G}_n^{k_1}\}_n$ and $\{\mathcal{G}_n^{k_2}\}_n$ ($k_1 \neq k_2$) satisfy the above, then $h_n^{k_1} = h_n^{k_2}$ and

$$\limsup_{n \rightarrow \infty} \left(\left| s_n^{k_1} - s_n^{k_2} \right| (1 + |\xi_n^{k_1}|) \right. \\ \left. + \left| y_n^{k_1} - y_n^{k_2} - 3 (s_n^{k_1} - s_n^{k_2}) (\xi_n^{k_1})^2 \right| \right) < +\infty$$

and

$$|\xi_n^{k_1} - \xi_n^{k_2}| \leq |\xi_n^{k_1} + \xi_n^j| + |\xi_n^{k_2} + \xi_n^j| \leq C < \infty.$$

These inequalities imply $\{\mathcal{G}_n^{k_1}\}_n$ and $\{\mathcal{G}_n^{k_2}\}_n$ are not orthogonal, a contradiction. A similar argument shows that orthogonality and space-time nonresonant property of $\{\mathcal{G}_n^j\}_n$ and $\{\mathcal{G}_n^k\}_n$ are equivalent as long as $\{\xi_n^j\}_n$ or $\{\xi_n^k\}_n$ is bounded.

The following is the main conclusion of space-time nonresonant property of two families.

Lemma 6.8. *Suppose that $\{\mathcal{G}_n\}_n, \{\tilde{\mathcal{G}}_n\}_n \subset G$ are space-time nonresonant families. Then, it holds for any $\psi, \tilde{\psi} \in \hat{L}^\alpha$ that*

$$\left\| \left[|\partial_x|^{\frac{1}{3\alpha}} e^{-t\partial_x^3} \mathcal{G}_n \psi \right] \overline{\left[|\partial_x|^{\frac{1}{3\alpha}} e^{-t\partial_x^3} \tilde{\mathcal{G}}_n \tilde{\psi} \right]} \right\|_{L_{t,x}^{\frac{3\alpha}{2}}} \rightarrow 0$$

as $n \rightarrow \infty$.

Proof. By a density argument, we assume that ψ and $\tilde{\psi}$ are Schwartz function with compact Fourier support. Let K be a compact set containing the Fourier support of ψ and $\tilde{\psi}$. Let $\Gamma_n = (h_n, \xi_n, y_n, s_n), \tilde{\Gamma}_n = (\tilde{h}_n, \tilde{\xi}_n, \tilde{y}_n, \tilde{s}_n) \in 2^{\mathbb{Z}} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ be parameters associated with $\{\mathcal{G}_n\}_n, \{\tilde{\mathcal{G}}_n\}_n \subset G$, respectively.

We first consider the case

$$\limsup_{n \rightarrow \infty} \left(\left| \log \frac{h_n}{\tilde{h}_n} \right| + \left| |\xi_n| - \frac{\tilde{h}_n}{h_n} |\tilde{\xi}_n| \right| \right) = +\infty.$$

Let us begin with the case when $\left| \log \frac{h_n}{\tilde{h}_n} \right| \rightarrow \infty$ as $n \rightarrow \infty$. The integrand equals

$$\iint |\xi|^{\frac{1}{3\alpha}} |\eta|^{\frac{1}{3\alpha}} e^{ix(\xi-\eta)+it(\xi^3-\eta^3)} \mathcal{F}[\mathcal{G}_n \psi](\xi) \overline{\mathcal{F}[\tilde{\mathcal{G}}_n \tilde{\psi}](\eta)} d\xi d\eta.$$

Then, by Hausdorff-Young inequality, our goal is to show

$$(6.6) \quad \iint_{K \times K} \frac{|\xi|^{\frac{1}{3\alpha-2}} |\eta|^{\frac{1}{3\alpha-2}} |\mathcal{F}[\mathcal{G}_n \psi](\xi)|^{\frac{3\alpha}{3\alpha-2}} |\mathcal{F}[\tilde{\mathcal{G}}_n \tilde{\psi}](\eta)|^{\frac{3\alpha}{3\alpha-2}}}{|\xi + \eta|^{\frac{2}{3\alpha-2}} |\xi - \eta|^{\frac{2}{3\alpha-2}}} d\xi d\eta \rightarrow 0$$

as $n \rightarrow \infty$. As in [42, Lemma 2.2], it holds that

$$\iint_{K \times K} \frac{|\xi|^{\frac{1}{3\alpha-2}} |\eta|^{\frac{1}{3\alpha-2}} |f(\xi)|^{\frac{3\alpha}{3\alpha-2}} |g(\eta)|^{\frac{3\alpha}{3\alpha-2}}}{|\xi + \eta|^{\frac{2}{3\alpha-2}} |\xi - \eta|^{\frac{2}{3\alpha-2}}} d\xi d\eta \leq C \|f\|_{L^{q_1}}^{\frac{3\alpha}{3\alpha-2}} \|g\|_{L^{q_2}}^{\frac{3\alpha}{3\alpha-2}},$$

where $q_1, q_2 \in (\frac{3\alpha}{3\alpha-2}, \frac{3\alpha}{3\alpha-4})$ satisfies

$$\frac{1}{q_1} + \frac{1}{q_2} = \frac{2}{\alpha'}.$$

Remark that

$$(6.7) \quad |\mathcal{F}[\mathcal{G}_n \psi](\xi)| = (\hat{D}(h_n) \hat{P}(\xi_n) |\mathcal{F}\psi|)(\xi) = h_n^{-\frac{1}{\alpha'}} |\mathcal{F}\psi| \left(\frac{\xi}{h_n} + \xi_n \right).$$

Similar formula holds for $|\mathcal{F}[\tilde{\mathcal{G}}_n \tilde{\psi}](\eta)|$. Therefore,

$$\|\mathcal{F}[\mathcal{G}_n \psi]\|_{L^{q_1}} \|\mathcal{F}[\tilde{\mathcal{G}}_n \tilde{\psi}]\|_{L^{q_2}} = \left(\frac{h_n}{\tilde{h}_n} \right)^{\frac{1}{q_1} - \frac{1}{\alpha'}} \|\mathcal{F}\psi\|_{L^{q_1}} \|\mathcal{F}\tilde{\psi}\|_{L^{q_2}}.$$

We have the desired smallness by taking $q_1 < \alpha'$ if $h_n/\tilde{h}_n \rightarrow 0$ as $n \rightarrow \infty$ and $q_1 > \alpha'$ otherwise.

We proceed to the case when $\limsup_{n \rightarrow \infty} \left| \log \frac{h_n}{\tilde{h}_n} \right| < \infty$. Changing notations and taking subsequence if necessary, we may suppose $\tilde{h}_n = h_n$. Recall that we need to show (6.6) as $n \rightarrow \infty$. By (6.7) and by change of variable, (LHS of (6.6))

$$\begin{aligned}
&= (h_n)^{\frac{4(\alpha-1)}{\alpha(3\alpha-2)}} \iint_{K \times K} \frac{|\xi - \xi_n|^{\frac{1}{3\alpha-2}} |\eta - \tilde{\xi}_n|^{\frac{1}{3\alpha-2}} |\mathcal{F}\psi(\xi)|^{\frac{3\alpha}{3\alpha-2}} |\mathcal{F}\tilde{\psi}(\eta)|^{\frac{3\alpha}{3\alpha-2}}}{|\xi + \eta - (\xi_n + \tilde{\xi}_n)|^{\frac{2}{3\alpha-2}} |\xi - \eta - (\xi_n - \tilde{\xi}_n)|^{\frac{2}{3\alpha-2}}} d\xi d\eta \\
&\leq (h_n)^{\frac{4(\alpha-1)}{\alpha(3\alpha-2)}} \left(\sup_{(\xi, \eta) \in K \times K} \frac{|\xi - \xi_n| |\eta - \tilde{\xi}_n|}{|\xi + \eta - (\xi_n + \tilde{\xi}_n)|^2 |\xi - \eta - (\xi_n - \tilde{\xi}_n)|^2} \right)^{\frac{1}{3\alpha-2}} \\
&\quad \times \iint_{K \times K} |\mathcal{F}\psi(\xi)|^{\frac{3\alpha}{3\alpha-2}} |\mathcal{F}\tilde{\psi}(\eta)|^{\frac{3\alpha}{3\alpha-2}} d\xi d\eta.
\end{aligned}$$

Notice that $||\xi_n| - |\tilde{\xi}_n|| \rightarrow \infty$ as $n \rightarrow \infty$ by orthogonality assumption. We have

$$\max(|\xi|, |\tilde{\xi}_n|) \leq \frac{1}{2}(|\xi_n + \tilde{\xi}_n| + |\xi_n - \tilde{\xi}_n|).$$

It therefore holds that

$$\frac{(1 + |\xi_n|)(1 + |\tilde{\xi}_n|)}{|\xi_n - \tilde{\xi}_n|^2 |\xi_n + \tilde{\xi}_n|^2} \leq \frac{1}{(\min(|\xi_n - \tilde{\xi}_n|, |\xi_n + \tilde{\xi}_n|))^2} = \frac{1}{\left| |\xi_n| - |\tilde{\xi}_n| \right|^2}$$

Hence, for large n ,

$$\begin{aligned}
&\sup_{(\xi, \eta) \in K \times K} \frac{|\xi - \xi_n| |\eta - \tilde{\xi}_n|}{|\xi + \eta - (\xi_n + \tilde{\xi}_n)|^2 |\xi - \eta - (\xi_n - \tilde{\xi}_n)|^2} \\
&\leq \sup_{(\xi, \eta) \in K \times K} \frac{2(C_K + |\xi_n|)(C_K + |\tilde{\xi}_n|)}{|\xi_n + \tilde{\xi}_n|^2 |\xi_n - \tilde{\xi}_n|^2} \leq \frac{C_K}{\left| |\xi_n| - |\tilde{\xi}_n| \right|^2}.
\end{aligned}$$

Therefore, we obtain the desired smallness (6.6).

Next, we assume that $h_n = \tilde{h}_n$, $\xi_n = \tilde{\xi}_n$, and

$$\limsup_{n \rightarrow \infty} \left(|s_n - \tilde{s}_n| (1 + |\xi_n|) + |y_n - \tilde{y}_n - 3(s_n - \tilde{s}_n)(\xi_n)^2| \right) = +\infty.$$

First, we further assume that $|\xi_n| \leq C$. Then, we may let $\xi_n \equiv 0$ by extracting subsequence and changing notations. In this case, the orthogonality implies $|s_n - \tilde{s}_n| + |y_n - \tilde{y}_n| \rightarrow \infty$ as $n \rightarrow \infty$. Since

$$|\partial_x|^{\frac{1}{3\alpha}} A(t) \mathcal{G}_n \psi = (h_n)^{\frac{4}{3\alpha}} [|\partial_x|^{\frac{1}{3\alpha}} A(\cdot) \psi] ((h_n)^3 t + s_n, h_n x - y_n)$$

and a similar formula for $|\partial_x|^{\frac{1}{3\alpha}} A(t) \tilde{\mathcal{G}}_n \tilde{\psi}$ hold, we see from change of variable that

$$\begin{aligned}
&\left\| [|\partial_x|^{\frac{1}{3\alpha}} e^{-t\partial_x^3} \mathcal{G}_n \psi] [\overline{|\partial_x|^{\frac{1}{3\alpha}} e^{-t\partial_x^3} \tilde{\mathcal{G}}_n \tilde{\psi}}] \right\|_{L_{t,x}^{\frac{3\alpha}{2}}} \\
&= \left\| [|\partial_x|^{\frac{1}{3\alpha}} A(\cdot) \psi](t, x) [\overline{|\partial_x|^{\frac{1}{3\alpha}} A(\cdot) \tilde{\psi}}(t - (s_n - \tilde{s}_n), x + (y_n - \tilde{y}_n))] \right\|_{L_{t,x}^{\frac{3\alpha}{2}}}.
\end{aligned}$$

It is obvious from $|\partial_x|^{\frac{1}{3\alpha}} A(t) \psi, |\partial_x|^{\frac{1}{3\alpha}} A(t) \tilde{\psi} \in L_{t,x}^{3\alpha}(\mathbb{R}^2)$ that the left hand side tends to zero as $|s_n - \tilde{s}_n| + |y_n - \tilde{y}_n| \rightarrow \infty$. We finally consider the case $|\xi_n| \rightarrow \infty$. Then,

$$|\partial_x|^{\frac{1}{3\alpha}} A(t) \mathcal{G}_n \psi = \frac{(h_n)^{\frac{4}{3\alpha}}}{\sqrt{2\pi}} \int |\xi|^{\frac{1}{3\alpha}} e^{i(h_n x - y_n)\xi + i((h_n)^3 t + s_n)\xi^3} \hat{\psi}(\xi + \xi_n) d\xi$$

$$\begin{aligned}
&= \frac{(h_n)^{\frac{4}{3\alpha}} |\xi_n|^{\frac{1}{3\alpha}}}{\sqrt{2\pi}} e^{-i((h_n)^3 t + s_n) \xi_n^3 - i(h_n x - y_n) \xi_n} \\
&\quad \times \int \left| 1 - \frac{\xi}{\xi_n} \right|^{\frac{1}{3\alpha}} e^{i \frac{t'}{3\xi_n} \xi^3} e^{-it' \xi^2 + ix' \xi} \hat{\psi}(\xi) d\xi,
\end{aligned}$$

where $t' = 3((h_n)^3 t + s_n) \xi_n$ and $x' = h_n x - y_n + 3((h_n)^3 t + s_n) \xi_n^2$. Hence, by change of variables,

$$\begin{aligned}
&\left\| |\partial_x|^{\frac{1}{3\alpha}} A(t) \mathcal{G}_n \psi \right\|_{L_{t,x}^{3\alpha}(\mathbb{R}^2)} \\
&= 3^{-\frac{1}{3\alpha}} \left\| \frac{1}{\sqrt{2\pi}} \int \left| 1 - \frac{\xi}{\xi_n} \right|^{\frac{1}{3\alpha}} e^{i \frac{t'}{3\xi_n} \xi^3} e^{-it' \xi^2 + ix' \xi} \hat{\psi}(\xi) d\xi \right\|_{L_{t',x'}^{3\alpha}(\mathbb{R}^2)}.
\end{aligned}$$

Since

$$\frac{1}{\sqrt{2\pi}} \int \left| 1 - \frac{\xi}{\xi_n} \right|^{\frac{1}{3\alpha}} e^{i \frac{t}{3\xi_n} \xi^3} e^{-it \xi^2 + ix \xi} \hat{\psi}(\xi) d\xi \rightarrow (S(\cdot) \psi)(t, x)$$

as $n \rightarrow \infty$ for any fixed $(t, x) \in \mathbb{R}^2$ and since

$$\left| \frac{1}{\sqrt{2\pi}} \int \left| 1 - \frac{\xi}{\xi_n} \right|^{\frac{1}{3\alpha}} e^{i \frac{t}{3\xi_n} \xi^3} e^{-it \xi^2 + ix \xi} \hat{\psi}(\xi) d\xi \right| \leq C_\psi (1 + |t|)^{-\frac{1}{4}} (1 + |x|)^{-\frac{1}{4}} \in L_{t,x}^{3\alpha}$$

as in Shao [51], the dominated convergence theorem gives us

$$\frac{1}{\sqrt{2\pi}} \int \left| 1 - \frac{\xi}{\xi_n} \right|^{\frac{1}{3\alpha}} e^{i \frac{t}{3\xi_n} \xi^3} e^{-it \xi^2 + ix \xi} \hat{\psi}(\xi) d\xi \rightarrow (S(\cdot) \psi)(t, x)$$

in $L_{t,x}^{3\alpha}(\mathbb{R}^2)$ as $n \rightarrow \infty$. Therefore, by Hölder's and Strichartz's estimates,

$$\begin{aligned}
&\left\| [|\partial_x|^{\frac{1}{3\alpha}} e^{-t\partial_x^3} \mathcal{G}_n \psi] [\overline{|\partial_x|^{\frac{1}{3\alpha}} e^{-t\partial_x^3} \tilde{\mathcal{G}}_n \tilde{\psi}}] \right\|_{L_{t,x}^{\frac{3\alpha}{2}}} \\
&\leq \left\| [S(\cdot) \psi](t', x') [\overline{S(\cdot) \tilde{\psi}}(t' + 3\xi_n(s_n - \tilde{s}_n), x' - (y_n - \tilde{y}_n - 3\xi_n^2(s_n - \tilde{s}_n)))] \right\|_{L_{t',x'}^{\frac{3\alpha}{2}}} \\
&\quad + o(1) \\
&= o(1)
\end{aligned}$$

with the help of orthogonality condition, which completes the proof. \square

Lemma 6.9. *Let $4/3 < \alpha < 2$ and $J \geq 1$. Let $\psi^j \in \hat{L}^\alpha$ ($1 \leq j \leq J$). Let $\{\mathcal{G}_n^j\}_n \subset G$ ($1 \leq j \leq J$) be mutually space-time nonresonant families. Then,*

$$\left\| \sum_{j=1}^J e^{-t\partial_x^3} \mathcal{G}_n^j \psi^j \right\|_{L(\mathbb{R})}^{3\alpha} \leq \sum_{j=1}^J \left\| e^{-t\partial_x^3} \mathcal{G}_n^j \psi^j \right\|_{L(\mathbb{R})}^{3\alpha} + o(1)$$

as $n \rightarrow \infty$.

Proof. Remark that $3\alpha \in (4, 6)$ is not necessarily an integer, and so we argue as in [1]. We consider the case $5 < 3\alpha < 6$, the other case is simpler. Set

$s := 3\alpha - 5 \in (0, 1)$ and $M := \max_j \|\psi^j\|_{\hat{L}^\alpha}$. For simplicity, we denote $U_n^j := |\partial_x|^{\frac{1}{3\alpha}} e^{-t\partial_x^3} \mathcal{G}_n^j \psi^j$. We have

$$\begin{aligned} \left\| \sum_{j=1}^J U_n^j \right\|_{L_{t,x}^{3\alpha}(\mathbb{R}^2)}^{3\alpha} &= \iint_{\mathbb{R}^2} \left| \sum_{j=1}^J U_n^j \right|^2 \left| \sum_{k=1}^J U_n^k \right|^2 \left| \sum_{l=1}^J U_n^l \right| \left| \sum_{m=1}^J U_n^m \right|^s dt dx \\ &\leq \sum_{j_1, j_2, k_1, k_2, l, m \in [1, J]} \iint_{\mathbb{R}^2} |U_n^{j_1} \overline{U_n^{j_2}}| |U_n^{k_1} \overline{U_n^{k_2}}| |U_n^l| |U_n^m|^s dt dx. \end{aligned}$$

Hence, it suffices to show that if $j_1 = j_2 = k_1 = k_2 = l = m$ fails then

$$(6.8) \quad A_n := \iint_{\mathbb{R}^2} |U_n^{j_1}| |U_n^{j_2}| |U_n^{k_1}| |U_n^{k_2}| |U_n^l| |U_n^m|^s dt dx = o(1)$$

as $n \rightarrow \infty$. If $j_1 \neq j_2$ then (6.4) and Lemma 6.8 yield

$$\begin{aligned} A_n &\leq \left\| U_n^{j_1} \overline{U_n^{j_2}} \right\|_{L_{t,x}^{\frac{3\alpha}{2}}} \left\| U_n^{k_1} \right\|_{L_{t,x}^{3\alpha}} \left\| U_n^{k_2} \right\|_{L_{t,x}^{3\alpha}} \left\| U_n^l \right\|_{L_{t,x}^{3\alpha}} \|U_n^m\|_{L_{t,x}^{3\alpha}}^s \\ &\leq CM^{3\alpha-2} \left\| U_n^{j_1} \overline{U_n^{j_2}} \right\|_{L_{t,x}^{\frac{3\alpha}{2}}} = o(1) \end{aligned}$$

as $n \rightarrow \infty$. The same argument shows (6.8) holds if $j_1 = j_2 = k_1 = k_2 = l$ fails. If $j_1 = j_2 = k_1 = k_2 = l \neq m$ then

$$A_n \leq \left\| U_n^l \overline{U_n^m} \right\|_{L_{t,x}^{\frac{3\alpha}{2}}}^s \left\| U_n^l \right\|_{L_{t,x}^{3\alpha}}^{3\alpha-2s} \leq CM^{3\alpha-2s} \left\| U_n^l \overline{U_n^m} \right\|_{L_{t,x}^{\frac{3\alpha}{2}}}^s = o(1)$$

as $n \rightarrow \infty$ as above. \square

6.3. Proof of Theorem 6.1. The proof is consists of three steps. The argument is very close to that in the mass-critical case $\alpha = 2$ such as [43, 8, 51].

Step 1 – Decomposition into a sum of scale pieces. Let us begin the proof of Theorem 6.1 with a decomposition of bounded sequence into some pieces of which Fourier transforms have mutually disjoint compact supports and are bounded.

Lemma 6.10. *Let $4/3 < \alpha < 2$ and $\sigma \in (\alpha', \frac{6\alpha}{3\alpha-2})$. Suppose that a bounded sequence $\{u_n\}_n \subset \hat{L}^\alpha$ satisfy $\|u_n\|_{\dot{M}_{2,\sigma}^\alpha} \leq M$. Then, for any $\epsilon > 0$, there exist a subsequence of $\{u_n\}$ which denoted still by $\{u_n\}$, a number J , $\{I_n^j = [h_n^j \xi_n^j, h_n^j (\xi_n^j + 1)]\}_n \subset \mathcal{D}$ ($1 \leq j \leq J$), $\{f_n^j\}_n \subset \hat{L}^\alpha$ ($1 \leq j \leq J$), and $q_n \in \hat{L}^\alpha$ such that*

$$\left| \log \frac{h_n^j}{h_n^k} \right| + \left| \xi_n^j - \frac{h_n^k}{h_n^j} \xi_n^k \right| \rightarrow \infty$$

as $n \rightarrow \infty$ for $1 \leq j < k \leq J$, and u_n is decomposed into

$$(6.9) \quad u_n = \sum_{j=1}^J f_n^j + q_n$$

for all $n \geq 1$. Moreover, it holds that

$$\|u_n\|_{\hat{M}_{2,\sigma}^\alpha}^\sigma \geq \sum_{j=1}^J \|f_n^j\|_{\hat{M}_{2,\sigma}^\alpha}^\sigma + \|q_n\|_{\hat{M}_{2,\sigma}^\alpha}^\sigma$$

for all $n \geq 1$ and

$$(6.10) \quad \limsup_{n \rightarrow \infty} \|q_n\|_{\hat{M}_{\frac{3\alpha}{2}, \frac{6\alpha}{3\alpha-2}}^\alpha} \leq \epsilon.$$

Further, there exists a bounded and compactly supported function F_j such that $\widehat{f_n^j}$ satisfies

$$(6.11) \quad |I_n^j|^{\frac{1}{\alpha'}} |\mathcal{F} f_n^j(h_n^j(\xi + \xi_n^j))| \leq F_j(\xi)$$

for any $n \geq 1$.

Remark 6.11. In the above decomposition, not only the number J but also f_n^j , q_n , and F^j depend on ϵ .

Proof. If $\limsup_{n \rightarrow \infty} \|u_n\|_{\hat{M}_{3\alpha/2, 6\alpha/(3\alpha-2)}^\alpha} \leq \epsilon$ then there is nothing to prove. Otherwise, we can extract a subsequence so that $\|u_n\|_{\hat{M}_{3\alpha/2, 6\alpha/(3\alpha-2)}^\alpha} > \epsilon$ for all n . By means of (6.4) and assumption, one sees that

$$\epsilon < C \|u_n\|_{\hat{M}_{2,\sigma}^\alpha}^\theta \|u_n\|_{\hat{M}_{\frac{3\alpha}{2}, \infty}^\alpha}^{1-\theta} \leq C M^\theta \|u_n\|_{\hat{M}_{\frac{3\alpha}{2}, \infty}^\alpha}^{1-\theta}.$$

Hence, by definition of $\hat{M}_{3\alpha/2, \infty}^\alpha$ norm, there exists an interval $I_n^1 := [h_n^1 \xi_n^1, h_n^1(\xi_n^1 + 1)] \in \mathcal{D}$ such that

$$(6.12) \quad \begin{aligned} \int_{I_n^1} |\hat{u}_n|^{\frac{3\alpha}{3\alpha-2}} d\xi &\geq C(M) \epsilon^{\frac{3\alpha}{(1-\theta)(3\alpha-2)}} |I_n^1|^{\frac{1}{3\alpha-2}} \\ &=: C_1 \epsilon^{\frac{3\alpha}{(1-\theta)(3\alpha-2)}} |I_n^1|^{\frac{1}{3\alpha-2}}, \end{aligned}$$

where C_1 is a constant depending only on α , σ , and M . On the other hand, for any $A > 0$, we have

$$(6.13) \quad \begin{aligned} \int_{I_n^1 \cap \{|\hat{u}_n| \geq A\}} |\hat{u}_n|^{\frac{3\alpha}{3\alpha-2}} d\xi &\leq A^{-\frac{3\alpha-4}{3\alpha-2}} \|\hat{u}_n\|_{L^2(I_n^1)}^2 \\ &\leq A^{-\frac{3\alpha-4}{3\alpha-2}} |I_n^1|^{\frac{2-\alpha}{\alpha}} \|u_n\|_{\hat{M}_{2,\sigma}^\alpha}^2 \\ &\leq M^2 A^{-\frac{3\alpha-4}{3\alpha-2}} |I_n^1|^{\frac{2-\alpha}{\alpha}}. \end{aligned}$$

We choose $A = (\frac{2M^2}{C_1})^{\frac{3\alpha-2}{3\alpha-4}} \epsilon^{-\frac{3\alpha}{(1-\theta)(3\alpha-4)}} |I_n^1|^{-\frac{1}{\alpha'}} =: C_\epsilon |I_n^1|^{-\frac{1}{\alpha'}}$ so that

$$M^2 A^{-\frac{3\alpha-4}{3\alpha-2}} |I_n^1|^{\frac{2-\alpha}{\alpha}} = \frac{C_1}{2} \epsilon^{\frac{3\alpha}{(1-\theta)(3\alpha-2)}} |I_n^1|^{\frac{1}{3\alpha-2}}.$$

From (6.12) and (6.13), we have

$$(6.14) \quad \begin{aligned} &\int_{I_n^1 \cap \{|\hat{u}_n| \leq C_\epsilon |I_n^1|^{-1/\alpha'}\}} |\hat{u}_n|^{\frac{3\alpha}{3\alpha-2}} d\xi \\ &\geq \int_{I_n^1} |\hat{u}_n|^{\frac{3\alpha}{3\alpha-2}} d\xi - \int_{I_n^1 \cap \{|\hat{u}_n| \geq A\}} |\hat{u}_n|^{\frac{3\alpha}{3\alpha-2}} d\xi \end{aligned}$$

$$\geq \frac{C_1}{2} \epsilon^{\frac{3\alpha}{(1-\theta)(3\alpha-2)}} |I_n^1|^{\frac{1}{3\alpha-2}}.$$

Hölder's inequality implies that

$$(6.15) \quad \int_{I_n^1 \cap \{|\hat{u}_n| \leq C_\epsilon |I_n^1|^{-1/\alpha'}\}} |\hat{u}_n|^{\frac{3\alpha}{3\alpha-2}} d\xi \leq \left(\int_{I_n^1 \cap \{|\hat{u}_n| \leq C_\epsilon |I_n^1|^{-1/\alpha'}\}} |\hat{u}_n|^2 d\xi \right)^{\frac{3\alpha}{2(3\alpha-2)}} |I_n^1|^{\frac{3\alpha-4}{2(3\alpha-2)}}.$$

Combining the inequalities (6.14) and (6.15), we reach to the estimate

$$(6.16) \quad |I_n^1|^{\frac{1}{2}-\frac{1}{\alpha}} \left(\int_{I_n^1 \cap \{|\hat{u}_n| \leq C_\epsilon |I_n^1|^{-1/\alpha'}\}} |\hat{u}_n|^2 d\xi \right)^{\frac{1}{2}} \geq \left(\frac{C_1}{2} \right)^{\frac{3\alpha-2}{3\alpha}} \epsilon^{\frac{1}{1-\theta}}.$$

We define v_n^1 by $\widehat{v_n^1} := \hat{u}_n \mathbf{1}_{I_n^1 \cap \{|\hat{u}_n| \leq C_\epsilon |I_n^1|^{-1/\alpha'}\}}$ and $q_n^1 := u_n - v_n^1$. Then,

(6.16) can be rewritten as $\|v_n^1\|_{\dot{M}_{2,\sigma}^\alpha} \geq C' \epsilon^{\frac{1}{1-\theta}}$. Further, we have

$$|I_n^1|^{\frac{1}{\alpha'}} \left| \widehat{v_n^1}(h_n^1(\xi + \xi_n^1)) \right| \leq C_\epsilon \mathbf{1}_{[0,1]}(\xi).$$

If $\limsup_{n \rightarrow \infty} \|q_n^1\|_{\dot{M}_{3\alpha/2, 6\alpha/(3\alpha-2)}^\alpha} \leq \epsilon$ then we have done. Otherwise, the same argument with u_n being replaced by q_n^1 enables us to define $I_n^2 := [h_n^2 \xi_n^2, h_n^2(\xi_n^2 + 1)]$, v_n^2 , and q_n^2 (up to subsequence). We repeat this argument and define $I_n^j := [h_n^j \xi_n^j, h_n^j(\xi_n^j + 1)]$, v_n^j , and q_n^j inductively. It is easy to see that

$$\|u_n\|_{\dot{M}_{2,\sigma}^\alpha}^\sigma \geq \sum_{j=1}^N \|v_n^j\|_{\dot{M}_{2,\sigma}^\alpha}^\sigma + \|q_n^N\|_{\dot{M}_{2,\sigma}^\alpha}^\sigma$$

since supports of $\{v_n^j\}_{1 \leq j \leq N}$ and q_n^N are disjoint in the Fourier side and since $\sigma > 2$. Since $\|v_n^j\|_{\dot{M}_{2,\sigma}^\alpha} \geq C' \epsilon^{\frac{1}{1-\theta}}$ for each j , together with an embedding $\|q_n^j\|_{\dot{M}_{3\alpha/2,\sigma}^\alpha} \leq \|q_n^j\|_{\dot{M}_{2,\sigma}^\alpha}$, we see that

$$\limsup_{n \rightarrow \infty} \|q_n^J\|_{\dot{M}_{\frac{3\alpha}{2}, \frac{6\alpha}{3\alpha-2}}^\alpha} \leq \epsilon$$

holds in $J = J(\epsilon)$ steps. Set $q_n := q_n^J$.

We reorganize v_n^j to obtain mutual asymptotic orthogonality. It is done as follows; We collect all $k \geq 2$ such that $|\log \frac{h_n^1}{h_n^k}| + \left| \xi_n^1 - \frac{h_n^k}{h_n^1} \xi_n^k \right|$ is bounded, and define $f_n^1 := v_n^1 + \sum_k v_n^k$. Since

$$\begin{aligned} & |I_n^1|^{\frac{1}{\alpha'}} \left| \widehat{f_n^1}(h_n^1(\xi + \xi_n^1)) \right| \\ &= \left(\frac{h_n^1}{h_n^k} \right)^{\frac{1}{\alpha'}} |I_n^k|^{\frac{1}{\alpha'}} \left| \widehat{v_n^k} \left(h_n^k \left[\frac{h_n^1}{h_n^k} \left\{ \xi + \left(\xi_n^1 - \frac{h_n^k}{h_n^1} \xi_n^k \right) \right\} + \xi_n^k \right] \right) \right| \\ &\leq C_\epsilon \left(\frac{h_n^1}{h_n^k} \right)^{\frac{1}{\alpha'}} \mathbf{1}_{[0,1]} \left(\frac{h_n^1}{h_n^k} \left\{ \xi + \left(\xi_n^1 - \frac{h_n^k}{h_n^1} \xi_n^k \right) \right\} \right), \end{aligned}$$

we see that $|I_n^1|^{\frac{1}{\alpha'}} \mathcal{F} f_n^1(h_n^1(\xi + \xi_n^1)) \leq F_1(\xi)$ for some bounded and compactly supported function F_1 . Similarly, we define f_n^j inductively. It is easy to

see that f_n^j possesses all properties we want. This completes the proof of Lemma 6.10. \square

Step 2 – Decomposition of each scale pieces. We next decompose functions obtained in the previous decomposition. This part is similar to [51, Lemmas 3.4 and 3.5].

Lemma 6.12. *Let $4/3 < \alpha < 2$ and $\sigma \in (\alpha', \frac{6\alpha}{3\alpha-2})$. Let $\xi_n \in \mathbb{R}$ be a given sequence. Let $F(\xi)$ be a nonnegative bounded function with compact support. Suppose that a sequence $R_n \in \hat{L}^\alpha$ satisfy*

$$(6.17) \quad |\widehat{R_n}(\xi)| \leq F(\xi).$$

Then, up to subsequence, there exist $\{\phi^a\}_a \subset \hat{L}^\alpha$ with $|\widehat{\phi^a}(\xi)| \leq F(\xi)$, $(y_n^a, s_n^a) \in \mathbb{R}^2$ with

$$\lim_{n \rightarrow \infty} (|s_n^a - s_n^{\tilde{a}}| + |\xi_n(s_n^a - s_n^{\tilde{a}})| + |y_n^a - y_n^{\tilde{a}} + 3(\xi_n)^2(s_n^a - s_n^{\tilde{a}})|) = \infty$$

for any $\tilde{a} \neq a$, and $\{R_n^a\}_{n,a} \subset \hat{L}^\alpha$ with $|\widehat{R_n^a}(\xi)| \leq F(\xi)$ such that

$$R_n(x) = \sum_{a=1}^A [P(\xi_n)^{-1} A(s_n^a) T(y_n^a) P(\xi_n) \phi^a](x) + R_n^A(x)$$

for any $A \geq 1$. Moreover, it holds that

$$(6.18) \quad \sum_{a=1}^A \|\psi^a\|_{\dot{M}_{2,r}^q}^r + \limsup_{n \rightarrow \infty} \|R_n^A\|_{\dot{M}_{2,r}^q}^r \leq \limsup_{n \rightarrow \infty} \|R_n\|_{\dot{M}_{2,r}^q}^r < \infty$$

for any $2 < q' < r < \infty$ and $A \geq 1$. Furthermore, we have

$$(6.19) \quad \limsup_{n \rightarrow \infty} \left\| e^{-t\partial_x^3} [P(\xi_n) R_n^A] \right\|_{L(\mathbb{R})} \rightarrow 0$$

as $A \rightarrow \infty$.

Remark 6.13. This lemma itself is a profile decomposition type result. The main differences between Lemma 6.12 and Theorem 6.2 are that (i) a strong assumption (6.17) is made; (ii) a family of deformations $\{\mathcal{G}_n\}_n \subset G$ is replaced by a family of deformations of the form $\{P(\xi_n)^{-1} A(s_n) T(y_n) P(\xi_n)\}_n$ with a *fixed* sequence $\{\xi_n\}_n \subset \mathbb{R}$ and some $\{(s_n, y_n)\} \subset \mathbb{R}^2$; and (iii) a remainder is small as in (6.19).

Proof. For a sequence $R = \{R_n\}_n \subset \hat{L}^\alpha$ with (6.17), we introduce $\mu_\xi(R)$ as follows:

$$\mu_\xi(R) := \sup \{ \ell(\phi) \mid \phi \in \mathcal{M}_\xi(R) \},$$

where

$$\mathcal{M}_\xi(R) := \left\{ \phi \in \hat{L}^\alpha \left| \begin{array}{l} \phi = \text{w-} \lim_{k \rightarrow \infty} P(\xi_{n_k})^{-1} T(y_{n_k})^{-1} A(s_{n_k})^{-1} P(\xi_{n_k}) R_{n_k}, \\ \exists (s_n, y_n) \in \mathbb{R}^2, \exists n_k : \text{subsequence} \end{array} \right. \right\}.$$

Then, we first show the decomposition with the smallness $\mu_\xi(R^A) \rightarrow 0$ as $A \rightarrow \infty$ instead of (6.19). This part is done as in Theorem 5.2. Before the proof, remark that $P^{-1}(\xi_n) A(s_n) T(y_n) P(\xi_n)$ is a multiplier-like deformation. Indeed, $\mathcal{F} P(\xi_n)^{-1} A(s_n) T(y_n) P(\xi_n) \mathcal{F}^{-1} = e^{-iy_n(\xi - \xi_n) + is_n(\xi - \xi_n)^3}$ is

phase-like. Therefore, if a sequence R satisfies (6.17) then any $\phi \in \mathcal{M}_\xi(R)$ satisfies (6.17).

Since the decomposition is shown in the essentially same way as in Theorem 5.2, we only treat the first step the decomposition, that is, the extraction of ϕ^1 and (s_n^1, y_n^1) under the assumption $\mu_\xi(R) > 0$. By assumption $\mu_\xi(R) > 0$ there exist $\phi^1 \in \mathcal{M}_\xi(R)$ such that $\ell(\phi^1) \geq \frac{1}{2}\mu_\xi(R)$. Further, by definition of $\mathcal{M}_\xi(R)$, there exists $\{(s_n^1, y_n^1)\}_n$ such that

$$P(\xi_n)^{-1}T(y_n^1)^{-1}A(s_n^1)^{-1}P(\xi_n)R_n \rightharpoonup \phi^1(x) \quad \text{weakly in } \hat{L}^\alpha$$

as $n \rightarrow \infty$ up to subsequence. Then,

$$\hat{P}(\xi_n)^{-1}\hat{T}(y_n^1)^{-1}\hat{A}(s_n^1)^{-1}\hat{P}(\xi_n)\widehat{R}_n(\xi) \rightharpoonup \widehat{\phi^1}(\xi) \quad \text{weakly in } L^{\alpha'}.$$

As mentioned above, ϕ^1 satisfies (6.17). We set

$$R_n^1 := R_n - P(\xi_n)^{-1}A(s_n^1)T(y_n^1)P(\xi_n)\phi^1.$$

Then,

$$P(\xi_n)^{-1}T(y_n^1)^{-1}A(s_n^1)^{-1}P(\xi_n)R_n^1 \rightharpoonup 0 \quad \text{weakly in } \hat{L}^\alpha$$

as $n \rightarrow \infty$. Since R_n and ψ^1 satisfy (6.17), we have $|\mathcal{F}R_n^1(\xi)| \leq 2F(\xi)$ for any n . The decouple inequality (6.18) is shown just as in Lemma 5.5². We note that $F \in L^{q'} \hookrightarrow M_{2,r}^{q'}$ for $2 < q' < r < \infty$ since F is bounded and compactly supported, and so that $\limsup_{n \rightarrow \infty} \|R_n\|_{\dot{M}_{2,r}^q} \leq \|F\|_{M_{2,r}^{q'}}$ is finite by means of the assumption (6.17). Repeat the above procedure to obtain all the results but (6.19).

Let us show (6.19). By extracting subsequence, we may suppose that either $\xi_n \rightarrow \xi_0 \in \mathbb{R}$ or $|\xi_n| \rightarrow \infty$ as $n \rightarrow \infty$. We first consider the case $\lim_{n \rightarrow \infty} \xi_n = \xi_0$. Define $\tilde{\alpha} \in (4/3, \alpha)$ by $2/\tilde{\alpha} = 1/\alpha + 3/4$. By Hölder's inequality,

$$\begin{aligned} (6.20) \quad & \left\| e^{-t\partial_x^3} [P(\xi_n)R_n^A] \right\|_{L(\mathbb{R})} \\ & \leq \left\| |\partial_x|^{\frac{1}{3\alpha}} e^{-t\partial_x^3} [P(\xi_n)R_n^A] \right\|_{L_{t,x}^{3\tilde{\alpha}}}^{\frac{\tilde{\alpha}}{\alpha}} \left\| |\partial_x|^{\frac{1}{3\alpha}} e^{-t\partial_x^3} [P(\xi_n)P_n^A] \right\|_{L_{t,x}^\infty}^{1-\frac{\tilde{\alpha}}{\alpha}}. \end{aligned}$$

Since $4/3 < \tilde{\alpha} < \alpha < 2$, we have

$$\begin{aligned} & \left\| |\partial_x|^{\frac{1}{3\alpha}} e^{-t\partial_x^3} [P(\xi_n)R_n^A] \right\|_{L_{t,x}^{3\tilde{\alpha}}} \\ & = \left\| |\partial_x|^{\frac{1}{3\alpha}} e^{-t\partial_x^3} |\partial_x|^{\frac{1}{3\alpha} - \frac{1}{3\tilde{\alpha}}} [P(\xi_n)R_n^A] \right\|_{L_{t,x}^{3\tilde{\alpha}}} \\ & \leq C \left\| |I|^{-\frac{1}{3\tilde{\alpha}}} \left\| |\xi|^{\frac{1}{3\alpha} - \frac{1}{3\tilde{\alpha}}} \mathcal{F}P(\xi_n)R_n^A \right\|_{L^{(\frac{3\tilde{\alpha}}{2})'(I)}} \right\|_{\ell_{\mathcal{D}}^{\frac{6\tilde{\alpha}}{3\tilde{\alpha}-2}}} \\ & \leq C \left\| |I|^{-\frac{1}{3\tilde{\alpha}}} \left\| \mathcal{F}P(\xi_n)R_n^A \right\|_{L^2(I)} \left\| |\xi|^{\frac{1}{3\alpha} - \frac{1}{3\tilde{\alpha}}} \right\|_{L^{\frac{6\tilde{\alpha}}{3\tilde{\alpha}-4}}(I)} \right\|_{\ell_{\mathcal{D}}^{\frac{6\tilde{\alpha}}{3\tilde{\alpha}-2}}} \\ & \leq C \left\| |I|^{\frac{1}{2} - (\frac{2}{3\tilde{\alpha}} + \frac{1}{4})} \left\| \mathcal{F}P(\xi_n)R_n^A \right\|_{L^2(I)} \right\|_{\ell_{\mathcal{D}}^{\frac{6\tilde{\alpha}}{3\tilde{\alpha}-2}}} \end{aligned}$$

²In fact, the proof is even easier than Lemma 5.5 because the considering deformation $P^{-1}(\xi_n)A(s_n)T(y_n)P(\xi_n)$ is multiplier-like.

$$\leq C \|P(\xi_n)R_n^A\|_{\hat{M}_{2, \frac{6\tilde{\alpha}}{3\tilde{\alpha}-2}}^{\frac{12\tilde{\alpha}}{3\tilde{\alpha}+8}}}.$$

Thus, thanks to (6.4) and (6.18) with $q = \frac{12\tilde{\alpha}}{3\tilde{\alpha}+8}$ and $r = \frac{6\tilde{\alpha}}{3\tilde{\alpha}-2}$, we see that

$$(6.21) \quad \sup_{A \geq 1} \limsup_{n \rightarrow \infty} \left\| |D|^{\frac{1}{3\alpha}} e^{-t\partial_x^3} [P(\xi_n)R_n^A] \right\|_{L_{t,x}^{3\tilde{\alpha}}} \\ \leq C \limsup_{n \rightarrow \infty} \|R_n^A\|_{\hat{M}_{2,r}^q} \leq C \|F\|_{L^{q'}} < \infty.$$

Let $\chi(x)$ be a smooth function such that $\hat{\chi}$ is compactly supported and $\hat{\chi} \equiv 1$ on $\text{supp } F$. Set $\chi_n(x) = P(\xi_n)\chi(x)$. Then, $\chi_n * (e^{-t\partial_x^3}[P(\xi_n)R_n^A]) = e^{-t\partial_x^3}[P(\xi_n)R_n^A]$. There exists (s_n, y_n) such that

$$(6.22) \quad \left\| |\partial_x|^{\frac{1}{3\alpha}} e^{-t\partial_x^3} [P(\xi_n)R_n^A] \right\|_{L_{t,x}^\infty} \\ \leq 2 |(\chi_n * |\partial_x|^{\frac{1}{3\alpha}} e^{-t\partial_x^3} [P(\xi_n)R_n^A])(-s_n, y_n)|.$$

A computation shows that

$$(\chi_n * |\partial_x|^{\frac{1}{3\alpha}} e^{-t\partial_x^3} [P(\xi_n)R_n^A])(-s_n, y_n) \\ = \int \chi_n(-x) |\partial_x|^{\frac{1}{3\alpha}} [T(y_n)^{-1} A(s_n)^{-1} P(\xi_n)R_n^A](x) dx \\ = \int \left(P(\xi_n) |\partial_x|^{\frac{1}{3\alpha}} [P(\xi_n)^{-1} \chi(-\cdot)] \right) (x) [P(\xi_n)^{-1} T(y_n)^{-1} A(s_n)^{-1} P(\xi_n)R_n^A](x) dx.$$

Since $\xi_n \rightarrow \xi_0 \in \mathbb{R}$ as $n \rightarrow \infty$, one verifies that $P(\xi_n) |\partial_x|^{\frac{1}{3\alpha}} [P(\xi_n)^{-1} \chi(-\cdot)]$ converges to some function strongly in $\hat{L}^{\alpha'}$ as $n \rightarrow \infty$. Hence, by definition of μ_ξ , we see that

$$\limsup_{n \rightarrow \infty} |(\chi_n * |\partial_x|^{\frac{1}{3\alpha}} e^{-t\partial_x^3} P(\xi_n)R_n^A)(-s_n, y_n)| \leq C_\chi \mu_\xi(R^A).$$

Combining the above inequality with (6.20), (6.21) and (6.22), we obtain (6.19) from $\mu_\xi(R^A) \rightarrow 0$ as $A \rightarrow \infty$.

If $\lim_{n \rightarrow \infty} |\xi_n| = \infty$ then it holds from Hörmander-Mikhlin multiplier theorem that

$$(6.23) \quad \left\| e^{-t\partial_x^3} [P(\xi_n)R_n^A] \right\|_{L(\mathbb{R})} \leq C |\xi_n|^{\frac{1}{3\alpha}} \left\| e^{-t\partial_x^3} [P(\xi_n)R_n^A] \right\|_{L_{t,x}^{3\alpha}(\mathbb{R}^2)}$$

for large n . Let $\tilde{\alpha} \in (4/3, \alpha)$ be the same number as in the previous case. Then,

$$(6.24) \quad |\xi_n|^{\frac{1}{3\alpha}} \left\| e^{-t\partial_x^3} [P(\xi_n)R_n^A] \right\|_{L_{t,x}^{3\alpha}} \\ \leq \left(|\xi_n|^{\frac{1}{3\alpha}} \left\| e^{-t\partial_x^3} [P(\xi_n)R_n^A] \right\|_{L_{t,x}^{3\tilde{\alpha}}} \right)^{\frac{\tilde{\alpha}}{\alpha}} \left\| e^{-t\partial_x^3} [P(\xi_n)R_n^A] \right\|_{L_{t,x}^\infty}^{1-\frac{\tilde{\alpha}}{\alpha}}.$$

Since support of $\mathcal{F}R_n^A$ is a subset of $\text{supp } F$, arguing as in Lemma 2.3, we have

$$\left\| e^{-t\partial_x^3} P(\xi_n)R_n^A \right\|_{L_{t,x}^{3\tilde{\alpha}}} = \left\| |\partial_x|^{\frac{1}{3\alpha}} e^{-t\partial_x^3} |\partial_x|^{-\frac{1}{3\alpha}} P(\xi_n)R_n^A \right\|_{L_{t,x}^{3\tilde{\alpha}}} \\ \leq C \left\| |I|^{\frac{1}{2}-\frac{1}{\alpha}} \left\| |\xi - \xi_n|^{-\frac{1}{3\alpha}} \mathcal{F}R_n^A \right\|_{L^2(I)} \right\|_{\ell_D^{\frac{6\tilde{\alpha}}{3\tilde{\alpha}-2}}}$$

$$\leq C_F |\xi_n|^{-\frac{1}{3\alpha}} \left\| |I|^{\frac{\tilde{\alpha}-2}{2\alpha}} \|\mathcal{F}R_n^A\|_{L^2(I)} \right\|_{\ell^{\frac{6\tilde{\alpha}}{3\alpha-2}}_{\mathcal{D}}}$$

for large n . Thus, it follows from (6.18) that

$$(6.25) \quad \sup_{A \geq 1} \limsup_{n \rightarrow \infty} |\xi_n|^{\frac{1}{3\alpha}} \left\| e^{-t\partial_x^3} P(\xi_n) R_n^A \right\|_{L_{t,x}^{3\tilde{\alpha}}} \leq C \limsup_{n \rightarrow \infty} \|R_n\|_{\dot{M}_{2, \frac{6\tilde{\alpha}}{3\alpha-2}}^{\tilde{\alpha}}} < \infty.$$

We estimate $L_{t,x}^\infty$ -norm. There exists (s_n, y_n) such that

$$\left\| e^{-t\partial_x^3} [P(\xi_n) P_n^A] \right\|_{L_{t,x}^\infty} \leq 2 |e^{-t\partial_x^3} [P(\xi_n) P_n^A](s_n, y_n)|.$$

The estimate for

$$(6.26) \quad \limsup_{n \rightarrow \infty} \left\| e^{-t\partial_x^3} [P(\xi_n) R_n^A] \right\|_{L_{t,x}^\infty} \rightarrow 0$$

as $A \rightarrow \infty$ is essentially the same as in the previous case. The difference is that we do not have to care about unboundedness of ξ_n because the derivative $|\partial_x|^{1/3\alpha}$ is removed. From (6.23), (6.24), (6.25) and (6.26), we have (6.19). This completes the proof of Lemma 6.12. \square

Step 3 –Completion of the proof of Theorem 6.1. Let $\{u_n\} \subset \hat{L}^\alpha$ be bounded sequence satisfying (6.1) and (6.2). Let $\varepsilon = \varepsilon(m, M) > 0$ to be chosen later. Let $J = J(\varepsilon) \geq 1$, $\{I_n^j = [h_n^j \xi_n^j, h_n^j(\xi_n^j + 1)]\}_n \subset \mathcal{D}$ ($1 \leq j \leq J$), $\{f_n^j\}_n \subset \hat{L}^\alpha$ ($1 \leq j \leq J$), and q_n be sequences given in Lemma 6.10. Set

$$\widehat{R_n^j}(\xi) := |h_n^j|^{\frac{1}{\alpha'}} \widehat{f_n^j}(h_n^j(\xi + \xi_n^j)).$$

Namely, $R_n^j = P(\xi_n^j)^{-1} D(h_n^j)^{-1} f_n^j$. Then, by means of (6.11), $\{R_n^j\}_n$ satisfies assumption of Lemma 6.12 with $\{\xi_n\}_n := \{\xi_n^j\}_n$ for each j . Then, thanks to Lemma 6.12, for every $1 \leq j \leq J$, there exists a family $\{\phi^{j,a}\}_a \subset \mathcal{M}_{\xi^j}(R^j)$, and a family $\{(y_n^{j,a}, s_n^{j,a})\}_{n,a} \in \mathbb{R} \times \mathbb{R}$ such that

$$R_n^j = \sum_{a=1}^A P(\xi_n^j)^{-1} A(s_n^{j,a}) T(y_n^{j,a}) P(\xi_n^j) \phi^{j,a} + R_n^{j,A}$$

with

$$\limsup_{n \rightarrow \infty} \left\| e^{-t\partial_x^3} [P(\xi_n^j) R_n^{j,A}] \right\|_{L(\mathbb{R})} \rightarrow 0$$

as $A \rightarrow \infty$ and that

$$\lim_{n \rightarrow \infty} (|s_n^{j,a} - s_n^{j,\tilde{a}}| + |\xi_n^j(s_n^{j,a} - s_n^{j,\tilde{a}})| + |y_n^{j,a} - y_n^{j,\tilde{a}} + 3(\xi_n^j)^2(s_n^{j,a} - s_n^{j,\tilde{a}})|) = \infty$$

for any $a \neq \tilde{a}$. Remark that

$$(6.27) \quad \begin{aligned} f_n^j &= D(h_n^j) P(\xi_n^j) R_n^j \\ &= \sum_{a=1}^A D(h_n^j) A(s_n^{j,a}) T(y_n^{j,a}) P(\xi_n^j) \phi^{j,a} + D(h_n^j) P(\xi_n^j) R_n^{j,A}. \end{aligned}$$

We choose $A = A(\varepsilon)$ so that

$$(6.28) \quad \limsup_{n \rightarrow \infty} \left\| e^{-t\partial_x^3} D(h_n^j) P(\xi_n^j) R_n^{j,A} \right\|_{L(\mathbb{R})} \leq \frac{\varepsilon}{J}$$

holds for any $1 \leq j \leq J$. Notice that this is possible by means of the scale invariance

$$\left\| e^{-t\partial_x^3} D(h_n^j) P(\xi_n^j) R_n^{j,A} \right\|_{L(\mathbb{R})} = \left\| e^{-t\partial_x^3} P(\xi_n^j) R_n^{j,A} \right\|_{L(\mathbb{R})}.$$

Let $r_n := \sum_{j=1}^J D(h_n^j) P(\xi_n^j) R_n^{j,A} + q_n$. By Lemma 6.10 (6.9) and (6.27), we have

$$(6.29) \quad u_n = \sum_{j=1}^J f_n^j + q_n = \sum_{j=1}^J \sum_{a=1}^A \mathcal{G}_n^{j,a} \phi^{j,a} + r_n,$$

where $\mathcal{G}_n^{j,a} := D(h_n^j) A(s_n^{j,a}) T(y_n^{j,a}) P(\xi_n^j)$. It is easy to see that $\{\mathcal{G}_n^{j,a}\}_n \subset G$ are pairwise orthogonal families. Recall that, for each (j, a) , the number of the pair (\tilde{j}, \tilde{a}) such that $\{\mathcal{G}_n^{j,a}\}_n$ and $\{\mathcal{G}_n^{\tilde{j}, \tilde{a}}\}_n$ are not space-time nonresonant is at most one (see, Remark 6.7). Let $\mathcal{A} = \mathcal{A}(\varepsilon) := \{(j, a) \mid 1 \leq j \leq J, 1 \leq a \leq A\}$. We divide \mathcal{A} into disjoint subsets \mathcal{A}_k ($1 \leq k \leq K$) so that

- (i) $1 \leq \#\mathcal{A}_k \leq 2$ is non-decreasing in k ;
- (ii) if $(j, a) \in \mathcal{A}_k$ and $(\tilde{j}, \tilde{a}) \in \mathcal{A}_{\tilde{k}}$, $k \neq \tilde{k}$, then $\{\mathcal{G}_n^{j,a}\}_n$ and $\{\mathcal{G}_n^{\tilde{j}, \tilde{a}}\}_n$ are space-time nonresonant;
- (iii) if $\#\mathcal{A}_k = 2$ and if $(j, a), (\tilde{j}, \tilde{a}) \in \mathcal{A}_k$, $(j, a) \neq (\tilde{j}, \tilde{a})$, then $\{\mathcal{G}_n^{j,a}\}_n$ and $\{\mathcal{G}_n^{\tilde{j}, \tilde{a}}\}_n$ are not space-time nonresonant.

Let K' be the number such that $\max\{k \mid \#\mathcal{A}_k = 1\}$. For $1 \leq k \leq K'$, we identify k and the unique pair $(j, a) \in \mathcal{A}_k$. Then, we have

$$u_n = \sum_{k=1}^{K'} \mathcal{G}_n^k \phi^k + \sum_{k=K'+1}^K \sum_{(j,a) \in \mathcal{A}_k} \mathcal{G}_n^{j,a} \phi^{j,a} + r_n.$$

By definition of r_n , (6.28) and Lemma 6.10 (6.10), we have

$$\begin{aligned} \left\| e^{-t\partial_x^3} u_n \right\|_{L(\mathbb{R})} &\leq \left\| e^{-t\partial_x^3} (u_n - r_n) \right\|_{L(\mathbb{R})} + \left\| e^{-t\partial_x^3} r_n \right\|_{L(\mathbb{R})} \\ &\leq \left\| e^{-t\partial_x^3} (u_n - r_n) \right\|_{L(\mathbb{R})} + C\varepsilon. \end{aligned}$$

Combining the above inequality with the argument used in the proof of Lemma 6.9, one can verify that

$$\begin{aligned} \left\| e^{-t\partial_x^3} u_n \right\|_{L(\mathbb{R})}^{3\alpha} &\leq \sum_{k=1}^{K'} \left\| e^{-t\partial_x^3} \mathcal{G}_n^k \phi^k \right\|_{L(\mathbb{R})}^{3\alpha} + \sum_{k=K'+1}^K \left\| \sum_{(j,a) \in \mathcal{A}_k} e^{-t\partial_x^3} \mathcal{G}_n^{j,a} \phi^{j,a} \right\|_{L(\mathbb{R})}^{3\alpha} \\ &\quad + C(1 + M^{3\alpha-1})\varepsilon + o(1) \end{aligned}$$

as $n \rightarrow \infty$. Further, by the triangle inequality,

$$\left\| \sum_{(j,a) \in \mathcal{A}_k} e^{-t\partial_x^3} \mathcal{G}_n^{j,a} \phi^{j,a} \right\|_{L(\mathbb{R})}^{3\alpha} \leq 2^{3\alpha} \sum_{(j,a) \in \mathcal{A}_k} \left\| e^{-t\partial_x^3} \mathcal{G}_n^{j,a} \phi^{j,a} \right\|_{L(\mathbb{R})}^{3\alpha}.$$

Combining the above estimates and going back to the notation (j, a) , one has

$$\left\| e^{-t\partial_x^3} u_n \right\|_{L(\mathbb{R})}^{3\alpha} \leq 2^{3\alpha} \sum_{(j,a) \in \mathcal{A}} \left\| e^{-t\partial_x^3} \mathcal{G}_n^{j,a} \phi^{j,a} \right\|_{L(\mathbb{R})}^{3\alpha} + C(1 + M^{3\alpha-1})\epsilon + o(1).$$

By (6.2), we can take $\epsilon = \epsilon(m, M)$ small and n large enough to get

$$C_\alpha m^{3\alpha} \leq \sum_{(j,a) \in \mathcal{A}} \left\| e^{-t\partial_x^3} \mathcal{G}_n^{j,a} \phi^{j,a} \right\|_{L(\mathbb{R})}^{3\alpha}.$$

By the refined Stein-Tomas inequality (Theorem 6.5 (6.4)) and Lemma 2.3,

$$\left\| e^{-t\partial_x^3} \mathcal{G}_n^{j,a} \phi^{j,a} \right\|_{L(\mathbb{R})} \leq C \left\| \mathcal{G}_n^{j,a} \phi^{j,a} \right\|_{\dot{M}_{2,\sigma}^\alpha} \leq C \left\| \phi^{j,a} \right\|_{\dot{M}_{2,\sigma}^\alpha}$$

for $\alpha' < \sigma < \frac{6\alpha}{3\alpha-2}$. Since $3\alpha > \sigma$, we have

$$\begin{aligned} C_\alpha m^{3\alpha} &\leq C \left(\sup_{j,a} \left\| e^{-t\partial_x^3} \mathcal{G}_n^{j,a} \phi^{j,a} \right\|_{L(\mathbb{R})} \right)^{3\alpha-\sigma} \sum_{(j,a) \in \mathcal{A}} \left\| \phi^{j,a} \right\|_{\dot{M}_{2,\sigma}^\alpha}^\sigma \\ &\leq C \left(\sup_{j,a} \left\| e^{-t\partial_x^3} \mathcal{G}_n^{j,a} \phi^{j,a} \right\|_{L(\mathbb{R})} \right)^{3\alpha-\sigma} M^\sigma. \end{aligned}$$

Thus, there exists (j_0, a_0) such that

$$(6.30) \quad \left\| e^{-t\partial_x^3} \mathcal{G}_n^{j_0, a_0} \phi^{j_0, a_0} \right\|_{L(\mathbb{R})} \geq C_\alpha \left(\frac{m^{3\alpha}}{M^\sigma} \right)^{\frac{1}{3\alpha-\sigma}}.$$

Now, up to subsequence, we have

$$(\mathcal{G}_n^{j_0, a_0})^{-1} u_n \rightharpoonup \phi^{j_0, a_0} + q =: \psi \quad \text{weakly in } \hat{L}^\alpha$$

as $n \rightarrow \infty$, where q is a weak limit of $(\mathcal{G}_n^{j_0, a_0})^{-1} q_n$. Indeed, by Lemma 6.10 (6.9), we have

$$u_n = \sum_{1 \leq j \leq J, j \neq j_0} f_n^j + f_n^{j_0} + q_n.$$

As in the proof of the first assertion of Lemma 5.4, one has $(\mathcal{G}_n^{j_0, a_0})^{-1} f_n^j \rightharpoonup 0$ weakly in \hat{L}^α as $n \rightarrow \infty$ for $j \neq j_0$. Further, $(\mathcal{G}_n^{j_0, a_0})^{-1} f_n^{j_0} \rightharpoonup \phi^{j_0, a_0}$ as $n \rightarrow \infty$. Therefore, we have the above limit. Then, one sees from Lemma 2.3 and Lemma 6.10 (6.10) that, for any bounded set $K \subset \mathbb{Z}^2$,

$$\begin{aligned} &\sum_{(j,k) \in K} \left(|\tau_k^j|^{-\frac{1}{3\alpha}} \left\| \mathcal{F} q \right\|_{L^{(\frac{3\alpha}{2})'}(\tau_k^j)} \right)^{\frac{6\alpha}{3\alpha-2}} \\ &\leq \limsup_{n \rightarrow \infty} \sum_{(j,k) \in K} \left(|\tau_k^j|^{-\frac{1}{3\alpha}} \left\| \mathcal{F} (\mathcal{G}_n^{j_0, a_0})^{-1} q_n \right\|_{L^{(\frac{3\alpha}{2})'}(\tau_k^j)} \right)^{\frac{6\alpha}{3\alpha-2}} \\ &\leq C \limsup_{n \rightarrow \infty} \|q_n\|_{\dot{M}_{\frac{3\alpha}{2}, \frac{6\alpha}{3\alpha-2}}^\alpha}^{\frac{6\alpha}{3\alpha-2}} \leq C \epsilon^{\frac{6\alpha}{3\alpha-2}}. \end{aligned}$$

Taking supremum in K , one obtains $\|q\|_{\hat{M}_{\frac{3\alpha}{2}, \frac{6\alpha}{3\alpha-2}}^\alpha} \leq C\epsilon$. Thanks to (6.4) and Lemma 2.3,

$$\left\| e^{-t\partial_x^3} \mathcal{G}_n^{j_0, a_0} q \right\|_{L(\mathbb{R})} \leq \left\| \mathcal{G}_n^{j_0, a_0} q \right\|_{\hat{M}_{\frac{3\alpha}{2}, \frac{6\alpha}{3\alpha-2}}^\alpha} \leq C\epsilon.$$

Finally, using Theorem 6.5 (6.4), Lemma 2.3, and (6.30), and choosing $\epsilon = \epsilon(m, M) > 0$ even smaller if necessary, we reach to the estimate

$$\begin{aligned} \|\psi\|_{\hat{M}_{\frac{3\alpha}{2}, \frac{6\alpha}{3\alpha-2}}^\alpha} &\geq C \left\| e^{-t\partial_x^3} \mathcal{G}_n^{j_0, a_0} \psi \right\|_{L(\mathbb{R})} \\ &\geq C \left\| e^{-t\partial_x^3} \mathcal{G}_n^{j_0, a_0} \phi^{j_0, a_0} \right\|_{L(\mathbb{R})} - C\epsilon \\ &\geq \frac{C}{2} \left(\frac{m^{3\alpha}}{M^\sigma} \right)^{\frac{1}{3\alpha-\sigma}} =: \beta(m, M), \end{aligned}$$

which completes the proof of Theorem 6.1.

6.4. Two improvements in special cases. We consider two improvements of Theorem 6.2, under some additional assumptions.

The first one is the case when functions in a sequence are real-valued. This is nothing but the case of Theorem 4.3.

Proof of Theorem 4.3. In addition to the assumption of Theorem 6.2, we assume that u_n is real valued. We already have a decomposition

$$u_n = \sum_{j=1}^J \mathcal{G}_n^j \psi^j + r_n^J$$

by Theorem 6.2. We now show that this is rewritten as in (4.3). Fix j . If $|\xi_n^j|$ is bounded in n then, extracting subsequence if necessary, $\xi_n^j \rightarrow \xi^j \in \mathbb{R}$ as $n \rightarrow \infty$. Then, $(\mathcal{G}_n^j)^{-1} u_n \rightharpoonup \psi^j$ in \hat{L}^α implies

$$A(s_n^j)^{-1} T(y_n^j)^{-1} D(h_n^j)^{-1} u_n \rightharpoonup P(\xi^j) \psi^j \quad \text{in } \hat{L}^\alpha.$$

Since the left hand side is real-valued, so is the right hand side. Then, $P(\xi^j) \psi^j = \text{Re}(P(\xi^j) \psi^j)$. Denoting $P(\xi^j) \psi^j$ again by ψ^j , we may let $\xi_n^j \equiv 0$.

Next consider the case $|\xi_n^j| \rightarrow \infty$ as $n \rightarrow \infty$. In particular, assume that $\xi_n^j \rightarrow \infty$ as $n \rightarrow \infty$. Then, the convergence $(\mathcal{G}_n^j)^{-1} u_n \rightharpoonup \overline{\psi^j}$ in \hat{L}^α implies

$$P(-\xi_n^j)^{-1} A(s_n^j)^{-1} T(y_n^j)^{-1} D(h_n^j)^{-1} u_n \rightharpoonup \overline{\psi^j} \quad \text{in } \hat{L}^\alpha.$$

Therefore, there exists k such that $\{\mathcal{G}_n^k\}_n$ is not orthogonal to the family $\{\overline{\mathcal{G}_n^j}\}_n := \{D(h_n^j) T(y_n^j) A(s_n^j) P(-\xi_n^j)\}_n$. Indeed, if not then the above convergence implies $\eta(r_n^J) \geq \|\overline{\psi^j}\|_{\hat{M}_{2, \sigma}^\alpha}$ for all $J \geq 1$, a contradiction. Then,

one can replace $\{\mathcal{G}_n^k\}_n$ and ψ^k by $\{\overline{\mathcal{G}_n^j}\}_n$ and $\overline{\psi^j}$, respectively. Denoting $\psi^j/2$ again by ψ^j , we obtain the result. This is the reason why $c_j = 2$ when $|\xi_n^j| \rightarrow \infty$ as $n \rightarrow \infty$. This completes the proof of Theorem 4.3.

The second one is exclusion of deformations $D(h)$ and $P(\xi)$ under uniform boundedness in a stronger topologies. This is the key for Theorem 1.5.

Proposition 6.14. (i) Under the assumptions in Theorem 6.2, assume in addition that $\{u_n\}_n$ is uniformly bounded in $\hat{L}^{\alpha_1} \cap \hat{L}^{\alpha_2}$ for some $1 < \alpha_1 < \alpha < \alpha_2 < \infty$. Then, the assertions of Theorem 6.2 hold with $h_n^j \equiv 1$. Furthermore, we have

$$\|\psi^j\|_{\hat{L}^\rho} \leq \limsup_{n \rightarrow \infty} \|u_n\|_{\hat{L}^\rho}$$

for all $j \geq 1$ and $\alpha_1 \leq \rho \leq \alpha_2$.

(ii) In addition to the assumption of Theorem 6.2, if $\{u_n\}_n$ is uniformly bounded in $\hat{L}^{\alpha_1} \cap \dot{H}^s$ for some $1 < \alpha_1 < \alpha$ and $s > 0$ then, the assertions of Theorem 6.2 hold with $h_n^j \equiv 1$, $\xi_n^j \equiv 0$. Furthermore, we have

$$\|\psi^j\|_{\dot{H}^s} \leq \limsup_{n \rightarrow \infty} \|u_n\|_{\dot{H}^s}$$

for all $j \geq 1$.

Proof. Suppose that u_n is uniformly bounded in $\hat{L}^{\alpha_1} \cap \hat{L}^{\alpha_2}$, $\alpha_1 < \alpha < \alpha_2$, and that

$$P(\xi_n)^{-1}A(s_n)^{-1}T(y_n)^{-1}D_\alpha(h_n)^{-1}u_n \rightharpoonup \psi \quad \text{in } \hat{L}^\alpha$$

for some $\psi \neq 0$. Then, for $g \in C^\infty$ such that \hat{g} has a compact support,

$$\begin{aligned} |(\psi, g)| &\leq 2 \left| \int (P(\xi_n)^{-1}A(s_n)^{-1}T(y_n)^{-1}D_\alpha(h_n)^{-1}u_n)(x) \overline{g(x)} dx \right| \\ &= 2 \left| \int u_n(x) \overline{(D_{\alpha'}(h_n)T(y_n)A(s_n)P(\xi_n)g)(x)} dx \right| \\ &\leq 2 \|u_n\|_{\hat{L}^{\alpha_1} \cap \hat{L}^{\alpha_2}} \|D_{\alpha'}(h_n)T(y_n)A(s_n)P(\xi_n)g\|_{\hat{L}^{\alpha'_1} + \hat{L}^{\alpha'_2}} \\ &\leq C \|D_{\alpha'}(h_n)T(y_n)A(s_n)P(\xi_n)g\|_{\hat{L}^{\alpha'_1} + \hat{L}^{\alpha'_2}} \end{aligned}$$

for large n . If $h_n \rightarrow 0$ as $n \rightarrow \infty$ then

$$\|D_{\alpha'}(h_n)T(y_n)A(s_n)P(\xi_n)g\|_{\hat{L}^{\alpha'_1}} = (h_n)^{\frac{1}{\alpha_1} - \frac{1}{\alpha}} \|g\|_{\hat{L}^{\alpha'_1}} \rightarrow 0$$

as $n \rightarrow \infty$. Similarly, if $h_n \rightarrow \infty$ as $n \rightarrow \infty$ then

$$\|D_{\alpha'}(h_n)T(y_n)A(s_n)P(\xi_n)g\|_{\hat{L}^{\alpha'_2}} = (h_n)^{\frac{1}{\alpha_2} - \frac{1}{\alpha}} \|g\|_{\hat{L}^{\alpha'_2}} \rightarrow 0$$

as $n \rightarrow \infty$. In the both cases, we have $\psi \equiv 0$, a contradiction. Thus, we conclude that $|\log h_n|$ is bounded. Extracting subsequence, we have $h_n \rightarrow h_0 > 0$ as $n \rightarrow \infty$. Then,

$$\begin{aligned} &P(h_n\xi_n)^{-1}A(s_n/(h_n)^3)^{-1}T(y_n/h_n)^{-1}u_n \\ &= D_\alpha(h_n)P(\xi_n)^{-1}A(s_n)^{-1}T(y_n)^{-1}D_\alpha(h_n)^{-1}u_n \rightharpoonup D_\alpha(h_0)\psi \quad \text{in } \hat{L}^p. \end{aligned}$$

Hence, denoting $(h_n\xi_n, s_n/(h_n)^3, y_n/h_n)$ and $D_\alpha(h_0)\psi$ again by (ξ_n, s_n, y_n) and ψ , respectively, we may let $h_n \equiv 1$. Under the new notation, we have

$$\|\psi\|_{\hat{L}^\rho} \leq \limsup_{n \rightarrow \infty} \|P(\xi_n)^{-1}A(s_n)^{-1}T(y_n)^{-1}u_n\|_{\hat{L}^\rho} = \limsup_{n \rightarrow \infty} \|u_n\|_{\hat{L}^\rho}.$$

for all $\alpha_1 \leq \rho \leq \alpha_2$.

Next, let us suppose that u_n is bounded in $\hat{L}^{\alpha_1} \cap \dot{H}^s$ ($\alpha_1 < \alpha$, $s > 0$). Note that this implies u_n is bounded in L^2 . Hence, the above argument

gives us $h_n^j \equiv 1$ for all $j \geq 1$. Let us show $\xi_n^j \equiv 0$ for all $j \geq 1$. For $g \in C^\infty$ such that \hat{g} has a compact support, we have

$$\begin{aligned} |(\psi, g)| &\leq 2 \left| \int (P(\xi_n)^{-1} A(s_n)^{-1} T(y_n)^{-1} u_n)(x) \overline{g(x)} dx \right| \\ &= 2 \left| \int u_n(x) \overline{(T(y_n) A(s_n) P(\xi_n) g)(x)} dx \right| \\ &\leq 2 \|u_n\|_{\dot{H}^s} \|T(y_n) A(s_n) P(\xi_n) g\|_{\dot{H}^{-s}} \leq C \|P(\xi_n) g\|_{\dot{H}^{-s}} \end{aligned}$$

for large n . If $|\xi_n| \rightarrow \infty$ as $n \rightarrow \infty$ then $\|P(\xi_n) g\|_{\dot{H}^{-s}} \rightarrow 0$ as $n \rightarrow \infty$. This gives us $\psi \equiv 0$, a contradiction. Hence, ξ_n is bounded. By extracting subsequence, $\xi_n \rightarrow \xi_0 \in \mathbb{R}$ as $n \rightarrow \infty$. Then,

$$A(s_n)^{-1} T(y_n)^{-1} u_n = P(\xi_n) P(\xi_n)^{-1} A(s_n)^{-1} T(y_n)^{-1} u_n \rightharpoonup P(\xi_0) \psi \quad \text{in } \hat{L}^p.$$

Thus, denoting $P(\xi_0) \psi$ again by ψ , we may let $\xi_n \equiv 0$ and we have the bound $\|\psi\|_{\dot{H}^s} \leq \limsup_{n \rightarrow \infty} \|u_n\|_{\dot{H}^s}$. \square

7. QUICK REVIEW ON WELL-POSEDNESS OF (NLS)

In this section, we briefly summarize well-posedness and stability results for (NLS) which are need to prove Theorem 4.4.

7.1. Well-posedness for NLS. We first consider well-posedness for (NLS) in \hat{L}^α -space and $\hat{M}_{\rho, \sigma}^\alpha$ -space. The initial value problem (NLS) is formulated as

$$(7.1) \quad v(t) = e^{-it\partial_x^2} v_0 + i\mu \int_0^t e^{-i(t-t')\partial_x^2} (|v|^{2\alpha} v)(t') dt'.$$

The following well-posedness result plays an important role in this subsection. This kind of result is well known (see [24, 57], for example).

Proposition 7.1. *Let $4/3 < \alpha < 4$. Then there exists a number $\delta > 0$ such that if a data $v_0 \in \mathcal{S}'$ and an interval $I \ni 0$ satisfies*

$$\left\| e^{-it\partial_x^2} v_0 \right\|_{L_{t,x}^{3\alpha}(I \times \mathbb{R})} \leq \delta$$

then there exists a unique solution $v(t)$ to (7.1) which satisfies

$$\|v\|_{L_{t,x}^{3\alpha}(I \times \mathbb{R})} \leq 2 \left\| e^{-it\partial_x^2} v_0 \right\|_{L_{t,x}^{3\alpha}(I \times \mathbb{R})}.$$

Further, the solution belongs to $L_t^p(I, L_x^q)$ for any $p, q \in (2, \infty)$ with $2/p + 1/q = 1/\alpha$.

Proof. Proposition 7.1 is an immediate consequence of the estimate

$$\|\Phi[v]\|_{L_{t,x}^{3\alpha}(I \times \mathbb{R})} \leq \left\| e^{-it\partial_x^2} v_0 \right\|_{L_{t,x}^{3\alpha}(I \times \mathbb{R})} + C \|v\|_{L_{t,x}^{3\alpha}(I \times \mathbb{R})}^{2\alpha+1}$$

for $4/3 < \alpha < 4$, where $\Phi[v]$ is the right hand side of (7.1). This inequality follows from Strichartz's estimate for non-admissible pairs (see Kato [24] or Lemma 7.3 (ii), below), and Hölder inequality. \square

Remark 7.2. Well-posedness of (7.1) in a space like $L_t^p(I; L_x^q)$ also holds for $\frac{1+\sqrt{17}}{4} < \alpha \leq 4/3$ if we allow the case $p \neq q$.

To prove well-posedness in \hat{L}^α -space, we show the following generalized Strichartz estimate for the Schrödinger equation.

Lemma 7.3. (i) (homogeneous estimates) Let I be an interval. Let (p, q) satisfy

$$0 \leq \frac{1}{p} < \frac{1}{4}, \quad 0 \leq \frac{1}{q} < \frac{1}{2} - \frac{1}{p}.$$

Then, for any $f \in \hat{L}^r$,

$$(7.2) \quad \left\| |\partial_x|^\tau e^{-it\partial_x^2} f \right\|_{L_x^p L_t^q(I)} \leq C \|f\|_{\hat{L}^r},$$

where

$$\frac{1}{r} = \frac{2}{p} + \frac{1}{q}, \quad \tau = -\frac{1}{p} + \frac{1}{q}.$$

and positive constant C depends only on r and s .

(ii) (inhomogeneous estimates) Let $4/3 < r < 4$ and let (p_j, q_j) ($j = 1, 2$) satisfy

$$0 \leq \frac{1}{p_j} < \frac{1}{4}, \quad 0 \leq \frac{1}{q_j} < \frac{1}{2} - \frac{1}{p_j}.$$

Then, the inequalities

$$(7.3) \quad \left\| \int_0^t e^{-i(t-t')\partial_x^2} F(t') dt' \right\|_{L_t^\infty(I; \hat{L}_x^r)} \leq C_1 \| |\partial_x|^{-\tau_2} F \|_{L_x^{p'_2} L_t^{q'_2}(I)},$$

and

$$(7.4) \quad \left\| |\partial_x|^{\tau_1} \int_0^t e^{-i(t-t')\partial_x^2} F(t') dt' \right\|_{L_x^{p_1} L_t^{q_1}(I)} \leq C_2 \| |\partial_x|^{-\tau_2} F \|_{L_x^{p'_2} L_t^{q'_2}(I)}$$

hold for any F satisfying $|\partial_x|^{-\tau_2} F \in L_x^{p'_2} L_t^{q'_2}$, where

$$\frac{1}{r} = \frac{2}{p_1} + \frac{1}{q_1}, \quad \tau_1 = -\frac{1}{p_1} + \frac{1}{q_1}$$

and

$$\frac{1}{r'} = \frac{2}{p_2} + \frac{1}{q_2}, \quad \tau_2 = -\frac{1}{p_2} + \frac{1}{q_2},$$

where the constant C_1 depends on r , τ_1 and I , and the constant C_2 depends on r , τ_1 , τ_2 and I .

Remark 7.4. Remark that we take a space-time norm of the form $L_x^p L_t^q$ in (7.2). This is why we gain derivative by $|\partial_x|^\tau$. Also remark that a similar estimate for a space-time norm of the form $L_t^p L_x^q$ is known in [22].

Proof. (7.2) is obtained by interpolating the Kato's smoothing effect [26, Theorem 4.1], the Kenig-Ruiz estimate [26, Theorem 2.5] and the Stein-Tomas estimate for the Schrödinger equation³

$$\left\| e^{-it\partial_x^2} f \right\|_{L_x^{3r} L_t^{3r}(I)} \leq C \|f\|_{\hat{L}^r}$$

for $r > 4/3$.

The inhomogeneous estimates (7.3) and (7.4) follows from the homogeneous estimate (7.2) and the Christ-Kiselev lemma by [44, Lemma 2]. \square

³This estimate goes back to [56]. We can prove this inequality by using the argument similar to [42, Lemma 2.2]

Inequality (7.2) and the following inequality yields the local well-posedness in \hat{L}^α and $\hat{M}_{\frac{3\alpha}{2}, 2(\frac{3\alpha}{2})'}^\alpha$, respectively;

Proposition 7.5. *Assume that $\alpha > 4/3$. Then,*

$$\left\| e^{-it\partial_x^2} f \right\|_{L_{t,x}^{3\alpha}(\mathbb{R} \times \mathbb{R})} \leq C \|f\|_{\hat{M}_{\frac{3\alpha}{2}, 2(\frac{3\alpha}{2})'}^\alpha}$$

holds for all $f \in \hat{M}_{\frac{3\alpha}{2}, 2(\frac{3\alpha}{2})'}^\alpha$. Further, the embedding $\hat{L}^\alpha \hookrightarrow \hat{M}_{\frac{3\alpha}{2}, 2(\frac{3\alpha}{2})'}^\alpha$ holds if $\alpha > 4/3$.

The inequality is shown as in Theorem 6.5 (see Remark B.2). The $\alpha = 2$ case is given in [1, 8]. Now, let us see how the well-posedness results are deduced. If either $v_0 \in \hat{M}_{\frac{3\alpha}{2}, 2(\frac{3\alpha}{2})'}^\alpha$ or $v_0 \in \hat{L}^\alpha$ then the above inequalities imply that $\|e^{-it\partial_x^2} v_0\|_{L_{t,x}^{3\alpha}(I \times \mathbb{R})} \leq \delta$ holds at least for small interval $I = I(v_0)$. Then, we obtain a solution $u(t)$ on I belonging to $L_{t,x}^{3\alpha}(I \times \mathbb{R})$ thanks to Proposition 7.1. Further, by applying (7.3), we see that

$$\left\| \Phi[v] - e^{-it\partial_x^2} v_0 \right\|_{L_t^\infty(I, \hat{L}_x^\alpha)} \leq C \|v\|_{L_{t,x}^{3\alpha}(I \times \mathbb{R})}^{2\alpha+1}.$$

Finally, the linear part $e^{-it\partial_x^2} v_0$ belongs to $C(\mathbb{R}; \hat{L}^\alpha)$ (resp. $C(\mathbb{R}; \hat{M}_{\frac{3\alpha}{2}, 2(\frac{3\alpha}{2})'}^\alpha)$) if $v_0 \in \hat{L}^\alpha$ (resp. $v_0 \in \hat{M}_{\frac{3\alpha}{2}, 2(\frac{3\alpha}{2})'}^\alpha$). Thus, we obtain the following.

Proposition 7.6 (Local well-posedness in \hat{L}^α and $\hat{M}_{\frac{3\alpha}{2}, 2(\frac{3\alpha}{2})'}^\alpha$). *Let $4/3 < \alpha < 4$.*

- (i) *For any $u_0 \in \hat{L}_x^\alpha$, there exists a unique solution $u(t) \in C(I; \hat{L}^\alpha)$.*
 - (ii) *For any $u_0 \in \hat{M}_{\frac{3\alpha}{2}, 2(\frac{3\alpha}{2})'}^\alpha$, there exists a unique solution $u(t) \in C(I; \hat{M}_{\frac{3\alpha}{2}, 2(\frac{3\alpha}{2})'}^\alpha)$.*
- Furthermore, $u(t) - e^{-it\partial_x^2} u_0 \in C(I, \hat{L}^\alpha)$ holds.*

Remark 7.7. It is obvious from the proof that a similar well-posedness result holds in all $\hat{M}_{\rho,\sigma}^\alpha$ space satisfying

$$\hat{L}^\alpha \hookrightarrow \hat{M}_{\rho,\sigma}^\alpha \hookrightarrow \hat{M}_{\frac{3\alpha}{2}, 2(\frac{3\alpha}{2})'}^\alpha.$$

Notice that the $\hat{M}_{2,\sigma}^\alpha$ space satisfies the above relation if $4/3 < \alpha < 2$ and $\alpha' < \sigma \leq 2(\frac{3\alpha}{2})' = 6\alpha/(3\alpha - 2)$. This is nothing but Theorem 1.11. On the other hand, the first assertion of the above proposition is Theorem 1.10.

As a corollary of this proposition, we obtain small data scattering in $\hat{M}_{\frac{3\alpha}{2}, 2(\frac{3\alpha}{2})'}^\alpha$.

Corollary 7.8. *Let $4/3 < \alpha < 4$ and . Assume that $v_0 \in \hat{L}^\alpha$ or $v_0 \in \hat{M}_{\frac{3\alpha}{2}, 2(\frac{3\alpha}{2})'}^\alpha$. There exists $\varepsilon > 0$ such that if $\hat{M}_{\frac{3\alpha}{2}, 2(\frac{3\alpha}{2})'}^\alpha < \varepsilon$ then $v_0 \in \mathcal{S}_{\text{NLS}}$.*

7.2. Persistence of regularity for NLS. Next we show the persistent property of solution to (NLS).

Lemma 7.9 (Persistence of $L_x^p L_t^q$ - and $L_t^p L_x^q$ -regularities). *Let $4/3 < \alpha < 4$ and $s \geq 0$. Let $\hat{t} \in \mathbb{R}$ and let I be a time interval containing \hat{t} . Assume*

that $v \in C(I; \hat{L}_x^\alpha(\mathbb{R}))$ is a solution to (NLS) satisfying $\|v\|_{L_{t,x}^{3\alpha}(I \times \mathbb{R})} \leq M$ for some M . Then, the following two assertions hold:

(i) If $|\partial_x|^s v(\hat{t}) \in \hat{L}^\alpha(\mathbb{R})$ then, for any

$$\tau \in \begin{cases} (\frac{1}{\alpha} - \frac{3}{4}, \frac{3}{2} - \frac{2}{\alpha}) & \text{if } \alpha < 2, \\ [-\frac{1}{2\alpha}, \frac{1}{\alpha}] & \text{if } \alpha \geq 2, \end{cases}$$

there exists a constant $C = C(\alpha, s, \tau, M)$ such that

$$(7.5) \quad \| |\partial_x|^s v \|_{L_t^\infty \hat{L}_x^\alpha(I \times \mathbb{R})} + \| |\partial_x|^{s+\tau} v \|_{L_x^p L_t^q(I)} \leq C \| |\partial_x|^s v(\hat{t}) \|_{\hat{L}^\alpha},$$

holds, where (p, q) satisfies

$$(7.6) \quad \frac{1}{\alpha} = \frac{2}{p} + \frac{1}{q}, \quad \tau = -\frac{1}{p} + \frac{1}{q}.$$

(ii) If $v(\hat{t}) \in \dot{H}^s(\mathbb{R})$ then, there exists $C = C(M)$ such that

$$(7.7) \quad \|v\|_{L_t^\infty(I; \dot{H}^s(\mathbb{R}))} + \| |\partial_x|^s v \|_{L_t^p(I; L_x^q(\mathbb{R}))} \leq C \|v(\hat{t})\|_{\dot{H}^s}.$$

holds, where (p, q) satisfies

$$(7.8) \quad 0 \leq \frac{1}{p} \leq \frac{1}{4}, \quad \frac{1}{2} = \frac{2}{p} + \frac{1}{q}.$$

Proof. Without loss of generality, we may assume that $\hat{t} = 0$ and $\inf I = 0$. We divide the time interval I into N subintervals such that

$$N \leq 1 + \left(\frac{M}{\eta} \right)^{3\alpha}, \quad I = \bigcup_{j=1}^N I_j, \quad I_j = [t_{j-1}, t_j]$$

with $\|v\|_{L_{t,x}^{3\alpha}(I_j \times \mathbb{R})} \leq \eta$ for any $1 \leq j \leq N$, where η is fixed later. Notice that such subdivision exists by the argument similar to the proof of Proposition 3.2.

We shall prove (7.5). To this end, we show

$$(7.9) \quad \| |\partial_x|^s v \|_{L_t^\infty \hat{L}_x^\alpha(I_j \times \mathbb{R})} + \| |\partial_x|^{s+\tau} v \|_{L_x^p L_t^q(I_j)} \leq C \| |\partial_x|^s v(t_j) \|_{\hat{L}^\alpha}$$

for any $1 \leq j \leq N$, where p, q satisfy (7.6). We first consider the case $j = 1$. By Lemma 7.3, we have

$$\begin{aligned} & \| |\partial_x|^s v \|_{L_t^\infty \hat{L}_x^\alpha(I_j \times \mathbb{R})} + \| |\partial_x|^{s+\tau} v \|_{L_x^p L_t^q(I_j)} + \| |\partial_x|^s v \|_{L_{t,x}^{3\alpha}(I_j \times \mathbb{R})} \\ & \leq C \| |\partial_x|^s v(0) \|_{\hat{L}^\alpha} + C \| |\partial_x|^s (|v|^{2\alpha} v) \|_{L_{t,x}^{\frac{3\alpha}{2\alpha+1}}(I_j)} \\ & \leq C \| |\partial_x|^s v(0) \|_{\hat{L}^\alpha} + C \|v\|_{L_{t,x}^{3\alpha}(I_j \times \mathbb{R})}^{2\alpha} \| |\partial_x|^s v \|_{L_{t,x}^{3\alpha}(I_j \times \mathbb{R})} \\ & \leq C \| |\partial_x|^s v(0) \|_{\hat{L}^\alpha} + C \eta^{2\alpha} \| |\partial_x|^s v \|_{L_{t,x}^{3\alpha}(I_j \times \mathbb{R})}. \end{aligned}$$

Choosing η sufficiently small so that $C\eta^{2\alpha} < 1$, we have (7.9) for $j = 1$. In particular, we obtain $\| |\partial_x|^s v(t_1) \|_{\hat{L}^\alpha} \leq C$. Hence a similar argument as above we have (7.9) for $j = 2$. Repeating this argument, we obtain (7.9) for any $1 \leq j \leq N$. Summing the inequalities (7.9) over all subintervals, we have (7.5).

The proof of (7.7) is done in a similar way. We use (usual) Strichartz's estimates instead. \square

7.3. Stability for NLS. In this section we consider the nonlinear Schrödinger equation with the perturbation:

$$(7.10) \quad \begin{cases} i\partial_t \tilde{v} - \partial_x^2 \tilde{v} = -\mu|\tilde{v}|^{2\alpha} \tilde{v} + e, & t, x \in \mathbb{R}, \\ \tilde{v}(\hat{t}, x) = \tilde{v}_0(x), & x \in \mathbb{R} \end{cases}$$

with the perturbation e small in a suitable sense and the initial data \tilde{v}_0 close to v_0 .

Proposition 7.10 (Long time stability for NLS). *Assume $4/3 < \alpha < 4$ and $\hat{t} \in \mathbb{R}$. Let I be a time interval containing \hat{t} and let \tilde{v} be a solution to (7.10) on $I \times \mathbb{R}$ for some function e . Assume that \tilde{v} satisfies*

$$\|\tilde{v}\|_{L_{t,x}^{3\alpha}(I \times \mathbb{R})} \leq M,$$

for some $M > 0$. Then there exists $\varepsilon_1 = \varepsilon_1(M) > 0$ such that if $v(\hat{t})$ and $\tilde{v}(\hat{t})$ satisfy

$$\|e^{-i(t-\hat{t})\partial_x^2}(v(\hat{t}) - \tilde{v}(\hat{t}))\|_{L_{t,x}^{3\alpha}(I \times \mathbb{R})} + \|e\|_{L_{t,x}^{\frac{3\alpha}{2\alpha+1}}(I \times \mathbb{R})} \leq \varepsilon$$

and $0 < \varepsilon < \varepsilon_1$, then there exists a solution $v \in L_{t,x}^{3\alpha}(I \times \mathbb{R})$ to (NLS) on $I \times \mathbb{R}$ satisfies

$$(7.11) \quad \|v - \tilde{v}\|_{L_{t,x}^{3\alpha}(I \times \mathbb{R})} \leq C\varepsilon,$$

$$(7.12) \quad \| |v|^{2\alpha} v - |\tilde{v}|^{2\alpha} \tilde{v} \|_{L_{t,x}^{\frac{3\alpha}{2\alpha+1}}(I \times \mathbb{R})} \leq C\varepsilon,$$

where the constant C depends on L . If, further, if $v(\hat{t}) - \tilde{v}(\hat{t}) \in \hat{L}^\alpha$ then

$$(7.13) \quad \|v - \tilde{v}\|_{L^\infty(I; \hat{L}_x^\alpha)} \leq \|v(\hat{t}) - \tilde{v}(\hat{t})\|_{\hat{L}_x^\alpha} + C\varepsilon.$$

Proof. The proof follows from the argument similar to the proof of Proposition 3.2 or as in [40]. We omit the detail. \square

8. EMBEDDING NLS INTO gKdV

In this section, we prove Theorem 4.4. As we mentioned in Introduction, we prove existence of a global solution u_n to (gKdV) by constructing approximating solution via the solution to the one dimensional nonlinear Schrödinger equation

$$(8.1) \quad i\partial_t v - \partial_x^2 v = -\mu C_0 |v|^{2\alpha} v,$$

where

$$C_0 = \frac{2\Gamma(\alpha + \frac{3}{2})}{3\sqrt{\pi}\Gamma(\alpha + 2)}.$$

With this constant, assumption (1.8) is written as

$$d_{+, \text{gKdV}} < 2^{1-\frac{1}{\sigma}} (C_0)^{-\frac{1}{2\alpha}} d_{\text{NLS}}.$$

Let v be a solution to (8.1) with the following conditions;

$$(8.2) \quad \begin{cases} v(T_0) = e^{-iT_0\partial_x^2} \phi & \text{if } |T_0| < \infty, \\ \lim_{t \rightarrow T_0} \|v(t) - e^{-it\partial_x^2} \phi\|_{\hat{L}_x^\alpha} = 0 & \text{if } T_0 = \pm\infty. \end{cases}$$

We now claim that v global and scatters for both time direction. Let us begin with the case $T_0 \in \mathbb{R}$. Remark that if v solves (8.1) then $(C_0)^{\frac{1}{2\alpha}}v$ solves (NLS). Hence, assumption of the theorem yields

$$\left\| (C_0)^{\frac{1}{2\alpha}} \phi \right\|_{\dot{M}_{2,\sigma}^\alpha} < 2^{1-\frac{1}{\sigma}} (C_0)^{\frac{1}{2\alpha}} d_{+, \text{gKdV}} < d_{\text{NLS}}.$$

Since $e^{-t\partial_x^3}$ is isometry in $\hat{M}_{2,\sigma}^\alpha$, $(C_0)^{\frac{1}{2\alpha}} e^{-T_0\partial_x^3} \phi \in \mathcal{S}_{+, \text{NLS}} \cap \mathcal{S}_{-, \text{NLS}}$ and so v scatters for both time direction. Next, if $T_0 = \infty$ then by definition v scatters for positive time direction and $\|v(t)\|_{\hat{M}_{2,\sigma}^\alpha} \rightarrow \|\phi\|_{\hat{M}_{2,\sigma}^\alpha}$ as $t \rightarrow \infty$. Therefore, we can take $T \in \mathbb{R}$ from maximal existence time of v so that $\|(C_0)^{1/2\alpha} v(T)\|_{\hat{M}_{2,\sigma}^\alpha} < d_{\text{NLS}}$. This implies that v scatters also for negative time. The case $T_0 = -\infty$ is handled in the same way. Thus,

$$v \in C(\mathbb{R}; \hat{L}_x^\alpha(\mathbb{R})) \cap L_{t,x}^{3\alpha}(\mathbb{R} \times \mathbb{R}).$$

We let $v_\pm \in \hat{L}_x^\alpha$ be scattering states such that

$$(8.3) \quad \lim_{T \rightarrow \infty} \|v(\pm T) - e^{\mp i T \partial_x^2} v_\pm\|_{\hat{L}_x^\alpha} = 0.$$

We further introduce v_n as a solution of (8.1) with

$$(8.4) \quad \begin{cases} v_n(T_0) = P_{|\xi| \leq \xi_n^{1/4}} e^{-iT_0 \partial_x^2} \phi & \text{if } |T_0| < \infty, \\ \lim_{t \rightarrow T_0} \|v_n(t) - P_{|\xi| \leq \xi_n^{1/4}} e^{-it \partial_x^2} \phi\|_{\hat{L}_x^\alpha} = 0 & \text{if } T_0 = \pm\infty, \end{cases}$$

where $P_{|\xi| \leq a} = \mathcal{F}^{-1} \varphi(\xi) \mathcal{F}$ with even bump function φ satisfying $\text{supp } \varphi \subset [-a, a]$. The long time stability for NLS (Proposition 7.10) yields

$$(8.5) \quad v_n \rightarrow v \text{ in } C(\mathbb{R}; \hat{L}_x^\alpha(\mathbb{R})) \cap L_{t,x}^{3\alpha}(\mathbb{R} \times \mathbb{R}).$$

In particular, v_n satisfies the uniform (in n) space-time bound

$$\|v_n\|_{L_{t,x}^{3\alpha}(\mathbb{R} \times \mathbb{R})} \leq C(\phi).$$

By the persistence of regularity for (NLS) (Lemma 7.9), we obtain

$$(8.6) \quad \||\partial_x|^s v_n\|_{L_t^\infty \hat{L}_x^\alpha} + \||\partial_x|^{s+\tau} v\|_{L_x^p L_t^q} \leq C \xi_n^{s/4}$$

for any $s \geq 0$, where $1/\alpha - 3/4 < \tau < 3/2 - 2/\alpha$ and (p, q) satisfies (7.6).

Further, since $\|P_{|\xi| \leq \xi_n^{1/4}} e^{-iT_0 \partial_x^2} \phi\|_{H_x^s} = O(\xi_n^{\frac{s}{4} - \frac{1}{8} + \frac{1}{4\alpha}})$ for any $s \geq 0$, it follows that

$$(8.7) \quad \begin{aligned} \||\partial_x|^s v_n\|_{L_t^p(\mathbb{R}, L_x^q)} &= O(\xi_n^{\frac{s}{4} - \frac{1}{8} + \frac{1}{4\alpha}}), \\ \||\partial_x|^s \partial_t v_n\|_{L_t^p(\mathbb{R}, L_x^q)} &= O(\xi_n^{\frac{s+2}{4} - \frac{1}{8} + \frac{1}{4\alpha}}) \end{aligned}$$

for any Schrödinger admissible pair (p, q) (i.e., (p, q) satisfies (7.8)) and $0 \leq s < 2\alpha$.

The convergence (8.5) gives us

$$(8.8) \quad \sup_n \|v_n\|_{L(|t|>T)} \rightarrow 0$$

as $T \rightarrow \infty$. Similarly, by (8.3) and (8.5),

$$(8.9) \quad \sup_n \|v_n(\pm T) - e^{\mp i T \partial_x^2} v_\pm\|_{\hat{L}_x^\alpha} \rightarrow 0$$

as $T \rightarrow \infty$.

Next, we construct a global solution u_n to (gKdV). As in [30], we introduce an approximate solution \tilde{u} to (gKdV):

$$(8.10) \quad \tilde{u}_n(t, x) := \begin{cases} Re[e^{-ix\xi_n - it\xi_n^3} v_n(-3\xi_n t, x + 3\xi_n^2 t)], & \text{if } |t| \leq \frac{T}{3\xi_n}, \\ e^{-(t - \frac{T}{3\xi_n})\partial_x^3} Re[e^{-ix\xi_n - \frac{i}{3}T\xi_n^2} v_n(-T, x + \xi_n T)], & \text{if } t > \frac{T}{3\xi_n}, \\ e^{-(t + \frac{T}{3\xi_n})\partial_x^3} Re[e^{-ix\xi_n + \frac{i}{3}T\xi_n^2} v_n(T, x - \xi_n T)], & \text{if } t < -\frac{T}{3\xi_n}, \end{cases}$$

where T is a large parameter independent of n which will be chosen later.

Lemma 8.1 (Space-time bound for \tilde{u}_n). *Assume $5/3 < \alpha < 2$. We have*

$$(8.11) \quad \|\tilde{u}_n\|_{L_t^\infty(\mathbb{R}; \hat{L}_x^\alpha)} + \|\tilde{u}_n\|_{L(\mathbb{R})} + \|\tilde{u}_n\|_{S(\mathbb{R})} \leq C,$$

where C is a positive constant independent of T and n .

Proof. We split the interval of integrals into $|t| > T/(3\xi_n)$ and $|t| \leq T/(3\xi_n)$. In the interval $|t| > T/(3\xi_n)$, each norms appearing in the left hand side of (8.11) are uniformly bounded in n by the homogenous estimate for Airy equation (Proposition 2.4) and the uniform space-time bound for v_n (8.6). In the interval $|t| \leq T/(3\xi_n)$, the space-time bound for v_n (8.6) yields

$$\|\tilde{u}_n\|_{L_t^\infty([- \frac{T}{3\xi_n}, \frac{T}{3\xi_n}]; \hat{L}_x^\alpha)} + \|\tilde{u}_n\|_{S(|t| \leq \frac{T}{3\xi_n})} \leq C,$$

where C is a positive constant independent of T and n . Combining the interpolation and (8.6), we see

$$\begin{aligned} \|\tilde{u}_n\|_{L(|t| \leq \frac{T}{3\xi_n})} &\leq C \|\tilde{u}_n\|_{L_{t,x}^{3\alpha}(|t| \leq T/3\xi_n)}^{1 - \frac{1}{3\alpha}} \|\partial_x \tilde{u}_n\|_{L_{t,x}^{3\alpha}(|t| \leq T/3\xi_n)}^{\frac{1}{3\alpha}} \\ &\leq C \xi_n^{-\frac{1}{3\alpha}(1 - \frac{1}{3\alpha}) + (1 - \frac{1}{3\alpha})\frac{1}{3\alpha}} = C. \end{aligned}$$

Collecting the above inequalities, we obtain (8.11). \square

Lemma 8.2. *Assume $5/3 < \alpha < 2$. Let $\phi \in \hat{L}_x^\alpha$ and let $\{\xi_n\}_{n \geq 1} \subset (0, \infty)$ such that $\xi_n \rightarrow \infty$ as $n \rightarrow \infty$. Then, we have*

$$(8.12) \quad \begin{aligned} \lim_{T \rightarrow \infty} \lim_{n \rightarrow \infty} \|e^{-t\partial_x^3} [e^{-ix\xi_n} e^{-iT\partial_x^2} \phi]\|_{L([0, \infty))} &= 0, \\ \lim_{T \rightarrow -\infty} \lim_{n \rightarrow \infty} \|e^{-t\partial_x^3} [e^{-ix\xi_n} e^{-iT\partial_x^2} \phi]\|_{L((-\infty, 0])} &= 0. \end{aligned}$$

Proof. By Proposition 2.7, it suffices to prove this lemma when ϕ satisfies $\hat{\phi} \in C_c^\infty(\mathbb{R})$. By the argument similar to [30], we obtain

$$\| |\partial_x|^{\frac{1}{3\alpha}} e^{-t\partial_x^3} [e^{-ix\xi_n} e^{-iT\partial_x^2} \phi] \|_{L_x^\infty} \leq C \frac{\xi_n^{\frac{1}{3\alpha}}}{(T - 3\xi_n t)^{1/2}} \|\phi\|_{L_x^1}$$

and

$$\| |\partial_x|^{\frac{1}{3\alpha}} e^{-t\partial_x^3} [e^{-ix\xi_n} e^{-iT\partial_x^2} \phi] \|_{L_x^2} \leq C \xi_n^{\frac{1}{3\alpha}} \|\phi\|_{H_x^{\frac{1}{3\alpha}}}.$$

Interpolating between the above two inequalities, we have

$$\| |\partial_x|^{\frac{1}{3\alpha}} e^{-t\partial_x^3} [e^{-ix\xi_n} e^{-iT\partial_x^2} \phi] \|_{L_x^{3\alpha}}^{3\alpha} \leq C(\phi) \frac{\xi_n}{(T - 3\xi_n t)^{\frac{3\alpha}{2}-1}}.$$

Integrating with respect to t variable, we obtain

$$\| e^{-t\partial_x^3} [e^{-ix\xi_n} e^{-iT\partial_x^2} \phi] \|_{L([0,\infty))}^{3\alpha} \leq C(\phi) T^{-\frac{3\alpha}{2}+1} \rightarrow 0,$$

as $T \rightarrow \infty$. This completes the proof. \square

Lemma 8.3 (Approximation of gKdV for large time). *Assume $5/3 < \alpha < 2$. Let \tilde{u}_n be given by (8.10). Then we have*

$$(8.13) \quad \lim_{T \rightarrow \infty} \limsup_{n \rightarrow \infty} \| |\partial_x|^{-1} \{ (\partial_t + \partial_x^3) \tilde{u}_n - \mu \partial_x (|\tilde{u}_n|^{2\alpha} \tilde{u}_n) \} \|_{N(|t| > \frac{T}{3\xi_n})} = 0.$$

Proof. Since \tilde{u}_n satisfies the Airy equation for $|t| > T/(3\xi_n)$, the linear part of (8.13) vanishes. Hence, we estimate $\| |\tilde{u}_n|^{2\alpha} \tilde{u}_n \|_{N(|t| > \frac{T}{3\xi_n})}$. We consider the case $t > T/(3\xi_n)$ only since the case $t < -T/(3\xi_n)$ being similar. Lemma 2.9 (i) implies

$$(8.14) \quad \| |\tilde{u}_n|^{2\alpha} \tilde{u}_n \|_{N([\frac{T}{3\xi_n}, \infty))} \leq C \| \tilde{u}_n \|_{S([\frac{T}{3\xi_n}, \infty))}^{2\alpha} \| \tilde{u}_n \|_{L([\frac{T}{3\xi_n}, \infty))}.$$

By (8.11), we have the bound $\| \tilde{u}_n \|_{S([\frac{T}{3\xi_n}, \infty))} \leq C$. On the other hand, Proposition 2.7 (2.4) yields

$$\begin{aligned} \| \tilde{u}_n \|_{L([\frac{T}{3\xi_n}, \infty))} &= \| e^{-t\partial_x^3} [e^{-ix\xi_n} v_n(T)] \|_{L([0, \infty))} \\ &\leq \| e^{-t\partial_x^3} [e^{-ix\xi_n} (v_n(T) - e^{-iT\partial_x^2} v_+)] \|_{L([0, \infty))} \\ &\quad + \| e^{-t\partial_x^3} [e^{-ix\xi_n} e^{-iT\partial_x^2} v_+] \|_{L([0, \infty))} \\ &\leq C \| v_n(T) - e^{-iT\partial_x^2} v_+ \|_{\dot{L}_x^\alpha} + \| e^{-t\partial_x^3} [e^{-ix\xi_n} e^{-iT\partial_x^2} v_+] \|_{L([0, \infty))}. \end{aligned}$$

This implies

$$\begin{aligned} \limsup_{n \rightarrow \infty} \| \tilde{u}_n \|_{L([\frac{T}{3\xi_n}, \infty))} &\leq C \sup_n \| v_n(T) - e^{-iT\partial_x^2} v_+ \|_{\dot{L}_x^\alpha} \\ &\quad + \limsup_{n \rightarrow \infty} \| e^{-t\partial_x^3} [e^{-ix\xi_n} e^{-iT\partial_x^2} v_+] \|_{L([0, \infty))} = 0 \end{aligned}$$

as $T \rightarrow \infty$ together with (8.9) and Lemma 8.2. Hence we obtain (8.13). \square

Next, we consider the approximation of gKdV in the middle interval $|t| \leq T/(3\xi_n)$. A direct calculation yields

$$(8.15) \quad \begin{aligned} (\partial_t + \partial_x^3) \tilde{u}_n &= 3\mu C_0 \xi_n \operatorname{Im} [e^{-ix\xi_n - it\xi_n^3} (|v_n|^{2\alpha} v_n) (-3\xi_n t, x + 3\xi_n^2 t)] \\ &\quad + \operatorname{Re} [e^{-ix\xi_n - it\xi_n^3} (\partial_x^3 v_n) (-3\xi_n t, x + 3\xi_n^2 t)] \end{aligned}$$

and

$$\begin{aligned} \mu \partial_x (|\tilde{u}_n|^{2\alpha} \tilde{u}_n) &= (2\alpha + 1) \mu \xi_n \operatorname{Re} [e^{-ix\xi_n - it\xi_n^3} v_n (-3\xi_n t, x + 3\xi_n^2 t)]^{2\alpha} \\ &\quad \times \operatorname{Im} [e^{-ix\xi_n - it\xi_n^3} v_n (-3\xi_n t, x + 3\xi_n^2 t)] \\ &\quad + (2\alpha + 1) \mu \operatorname{Re} [e^{-ix\xi_n - it\xi_n^3} v_n (-3\xi_n t, x + 3\xi_n^2 t)]^{2\alpha} \\ &\quad \times \operatorname{Re} [e^{-ix\xi_n - it\xi_n^3} (\partial_x v_n) (-3\xi_n t, x + 3\xi_n^2 t)]. \end{aligned}$$

Notice that

$$\begin{aligned}
& |Re[e^{-ix\xi_n - it\xi_n^3} v_n(-3\xi_n t, x + 3\xi_n^2 t)]|^{2\alpha} Im[e^{-ix\xi_n - it\xi_n^3} v_n(-3\xi_n t, x + 3\xi_n^2 t)] \\
&= (|v_n|^{2\alpha+1})(-3\xi_n t, x + 3\xi_n^2 t) |Re[e^{-ix\xi_n - it\xi_n^3 + iArgv_n}]|^{2\alpha} \\
&\quad \times Im[e^{-ix\xi_n - it\xi_n^3 + iArgv_n}] \\
&= (|v_n|^{2\alpha+1})(-3\xi_n t, x + 3\xi_n^2 t) \sum_{k=1}^{\infty} C_k Im[e^{-ik(x\xi_n + t\xi_n^3 - Argv_n)}] \\
&= C_1 Im[e^{-ix\xi_n - it\xi_n^3} (|v_n|^{2\alpha} v_n)(-3\xi_n t, x + 3\xi_n^2 t)] \\
&\quad + \sum_{k=2}^{\infty} C_k Im[e^{-ik(x\xi_n + t\xi_n^3)} (|v_n|^{2\alpha+1-k} v_n^k)(-3\xi_n t, x + 3\xi_n^2 t)],
\end{aligned}$$

where $Argv_n = Argv_n(-3\xi_n t, x + 3\xi_n^2 t)$ and C_k is a k -th Fourier-sin coefficients for an odd function $f(\theta) = |\cos \theta|^{2\alpha} \sin \theta$, i.e., C_k is the constant appearing in the expansion

$$|\cos \theta|^{2\alpha} \sin \theta = \sum_{k=1}^{\infty} C_k \sin(k\theta).$$

Or equivalently,

$$C_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin(k\theta) d\theta.$$

An elementary computation shows that

$$C_1 = \frac{2\Gamma(\alpha + \frac{1}{2})\Gamma(\frac{3}{2})}{\pi\Gamma(\alpha + 2)} = \frac{2\Gamma(\alpha + \frac{3}{2})}{\sqrt{\pi}(2\alpha + 1)\Gamma(\alpha + 2)} = \frac{3}{2\alpha + 1} C_0.$$

Then we have

$$\begin{aligned}
(8.16) \quad & (\partial_t + \partial_x^3) \tilde{u}_n - \mu \partial_x (|\tilde{u}_n|^{2\alpha} \tilde{u}_n) \\
&= Re[e^{-ix\xi_n - it\xi_n^3} (\partial_x^3 v_n)(-3\xi_n t, x + 3\xi_n^2 t)] \\
&\quad - (2\alpha + 1) \mu |Re[e^{-ix\xi_n - it\xi_n^3} v_n(-3\xi_n t, x + 3\xi_n^2 t)]|^{2\alpha} \\
&\quad \times Re[e^{-ix\xi_n - it\xi_n^3} (\partial_x v_n)(-3\xi_n t, x + 3\xi_n^2 t)] \\
&\quad - (2\alpha + 1) \mu \xi_n \sum_{k=2}^{\infty} C_k Im[e^{-ik(x\xi_n + t\xi_n^3)} (|v_n|^{2\alpha+1-k} v_n^k)(-3\xi_n t, x + 3\xi_n^2 t)] \\
&=: R_n^1 + R_n^2 + R_n^3.
\end{aligned}$$

To evaluate the right hand side of (8.16), we introduce a function e_n defined by

$$(8.17) \quad \begin{cases} (\partial_t + \partial_x^3) e_n = R_n^1 + R_n^2 + R_n^3, \\ e_n(0, x) = 0. \end{cases}$$

Set $e_n =: e_{n,1} + e_{n,2}$, where

$$\begin{aligned}
e_{n,1} &= (2\alpha + 1) \mu \xi_n^{-2} \\
&\quad \times \sum_{k=2}^{\infty} C_k Im \left[e^{-ikx\xi_n} \frac{e^{-ikt\xi_n^3} - e^{-ik^3 t\xi_n^3}}{i(k - k^3)} (|v_n|^{2\alpha+1-k} v_n^k)(-3\xi_n t, x + 3\xi_n^2 t) \right].
\end{aligned}$$

A direct calculation yields

$$\begin{aligned} (\partial_t + \partial_x^3)e_{n,1} &= R_n^3 + R_n^4, & e_{n,1}(0, x) &= 0, \\ (\partial_t + \partial_x^3)e_{n,2} &= R_n^1 + R_n^2 - R_n^4, & e_{n,2}(0, x) &= 0 \end{aligned}$$

where R_n^4 is given by

$$R_n^4 = \sum_{\ell=1}^4 \sum_{k=2}^{\infty} \text{Im} \left[G_n^{\ell,k}(-3\xi_n t, x + 3\xi_n^2 t) (e^{-ikt\xi_n^3} - e^{-ik^3 t\xi_n^3}) e^{-ikx\xi_n} \right]$$

with

$$\begin{aligned} G_n^{1,k}(t, x) &= 3(2\alpha + 1)\mu \frac{C_k}{ik} \partial_x (|v_n|^{2\alpha+1-k} v_n^k)(t, x), \\ G_n^{2,k}(t, x) &= -3(2\alpha + 1)\mu \frac{C_k}{1-k^2} \xi_n^{-1} \partial_x^2 (|v_n|^{2\alpha+1-k} v_n^k)(t, x), \\ G_n^{3,k}(t, x) &= -3(2\alpha + 1)\mu \frac{C_k}{i(k-k^3)} \xi_n^{-1} \partial_t (|v_n|^{2\alpha+1-k} v_n^k)(t, x), \\ G_n^{4,k}(t, x) &= (2\alpha + 1)\mu \frac{C_k}{i(k-k^3)} \xi_n^{-2} \partial_x^3 (|v_n|^{2\alpha+1-k} v_n^k)(t, x). \end{aligned}$$

Lemma 8.4 (Error control). *Fix $T > 0$. Let e_n be a solution to (8.17).*

Then,

$$(8.18) \quad \lim_{n \rightarrow \infty} \left(\|e_n\|_{L_t^\infty \hat{L}_x^\alpha([- \frac{T}{3\xi_n}, \frac{T}{3\xi_n}])} + \|e_n\|_{L([- \frac{T}{3\xi_n}, \frac{T}{3\xi_n}])} + \|e_n\|_{S([- \frac{T}{3\xi_n}, \frac{T}{3\xi_n}])} \right) = 0.$$

Proof. By the definition of $e_{n,1}$, we have

$$\begin{aligned} &\|e_{n,1}\|_{L_t^\infty \hat{L}_x^\alpha([- \frac{T}{3\xi_n}, \frac{T}{3\xi_n}])} \\ &\leq C \xi_n^{-2} \sum_{k \geq 2} \frac{|C_k|}{k^3} \|(|v_n|^{2\alpha+1-k} v_n^k)(3\xi_n t, x + 3\xi_n^2 t)\|_{L_t^\infty \hat{L}_x^\alpha([- \frac{T}{3\xi_n}, \frac{T}{3\xi_n}])}. \end{aligned}$$

Since $L^\alpha \hookrightarrow \hat{L}^\alpha$ and $\dot{H}^{\frac{1}{2} - \frac{1}{\alpha(2\alpha+1)}} \hookrightarrow L^{\alpha(2\alpha+1)}$ for $1 < \alpha \leq 2$, we see from (8.7) that

$$\begin{aligned} &\|(|v_n|^{2\alpha+1-k} v_n^k)(-3\xi_n t, x + 3\xi_n^2 t)\|_{L_t^\infty \hat{L}_x^\alpha([- \frac{T}{3\xi_n}, \frac{T}{3\xi_n}])} \\ &= C \|(|v_n|^{2\alpha+1-k} v_n^k)(t, x)\|_{L_t^\infty \hat{L}_x^\alpha([-T, T])} \\ &\leq C \| |v_n|^{2\alpha+1-k} v_n^k \|_{L_t^\infty L_x^\alpha([-T, T])} \\ &= C \|v_n\|_{L_t^\infty L_x^{\alpha(2\alpha+1)}([-T, T])}^{2\alpha+1} \\ &\leq C \|v_n\|_{L_t^\infty H_x^{\frac{1}{2} - \frac{1}{\alpha(2\alpha+1)}}}^{2\alpha+1} \\ &\leq C \xi_n^{\frac{1}{2}}, \end{aligned}$$

which implies

$$(8.19) \quad \|e_{n,1}\|_{L_t^\infty \hat{L}_x^\alpha([- \frac{T}{3\xi_n}, \frac{T}{3\xi_n}])} \leq C \xi_n^{-\frac{3}{2}} \sum_{k=2}^{\infty} \frac{|C_k|}{k^3} \leq C \xi_n^{-\frac{3}{2}} \rightarrow 0$$

as $n \rightarrow \infty$.

Next we evaluate the L -norm of $e_{n,1}$. An interpolation shows

$$\|e_{n,1}\|_{L([- \frac{T}{3\xi_n}, \frac{T}{3\xi_n}])} \leq \|e_{n,1}\|_{L_{t,x}^{3\alpha}([- \frac{T}{3\xi_n}, \frac{T}{3\xi_n}])}^{1-\frac{1}{3\alpha}} \|\partial_x e_{n,1}\|_{L_{t,x}^{3\alpha}([- \frac{T}{3\xi_n}, \frac{T}{3\xi_n}])}^{\frac{1}{3\alpha}}.$$

By the definition of $e_{n,1}$, we see

$$\begin{aligned} & \|\partial_x e_{n,1}\|_{L_{t,x}^{3\alpha}([- \frac{T}{3\xi_n}, \frac{T}{3\xi_n}])} \\ & \leq C\xi_n^{-1} \sum_{k=2}^{\infty} \frac{|C_k|}{k^2} \|(|v_n|^{2\alpha+1-k} v_n^k)(-3\xi_n t, x + 3\xi_n^2 t)\|_{L_{t,x}^{3\alpha}([- \frac{T}{3\xi_n}, \frac{T}{3\xi_n}])} \\ & \quad + C\xi_n^{-2} \sum_{k=2}^{\infty} \frac{|C_k|}{k^3} \|\partial_x(|v_n|^{2\alpha+1-k} v_n^k)(-3\xi_n t, x + 3\xi_n^2 t)\|_{L_{t,x}^{3\alpha}([- \frac{T}{3\xi_n}, \frac{T}{3\xi_n}])}. \end{aligned}$$

Change of variables, the embedding $\dot{W}_x^{\frac{1}{2}-\frac{1}{\alpha(2\alpha+1)}, \frac{6\alpha(2\alpha+1)}{6\alpha^2+3\alpha-4}} \hookrightarrow L_x^{3\alpha(2\alpha+1)}$ and (8.7) yield

$$\begin{aligned} & \|\partial_x^j(|v_n|^{2\alpha+1-k} v_n^k)(-3\xi_n t, x + 3\xi_n^2 t)\|_{L_{t,x}^{3\alpha}([- \frac{T}{3\xi_n}, \frac{T}{3\xi_n}])} \\ & \leq \|\partial_x^j(|v_n|^{2\alpha+1-k} v_n^k)(t, x)\|_{L_{t,x}^{3\alpha}([-T, T])} \\ & \leq \|v_n\|_{L_{t,x}^{3\alpha(2\alpha+1)}([-T, T])}^{2\alpha} \|\partial_x^j v_n\|_{L_{t,x}^{3\alpha(2\alpha+1)}([-T, T])} \\ & \leq \|\partial_x\|^{\frac{1}{2}-\frac{1}{\alpha(2\alpha+1)}} v_n^{2\alpha} \Big|_{L_t^{3\alpha(2\alpha+1)} L_x^{\frac{6\alpha(2\alpha+1)}{6\alpha^2+3\alpha-4}}([-T, T])} \\ & \quad \times \|\partial_x\|^{j+\frac{1}{2}-\frac{1}{\alpha(2\alpha+1)}} v_n \Big|_{L_{t,x}^{3\alpha(2\alpha+1)} L_x^{\frac{6\alpha(2\alpha+1)}{6\alpha^2+3\alpha-4}}([-T, T])} \\ & \leq C\xi_n^{-\frac{1}{3\alpha}+\frac{1}{2}+\frac{j}{4}} \end{aligned}$$

for $j = 0, 1$. Hence we obtain

$$\begin{aligned} \|\partial_x e_{n,1}\|_{L_{t,x}^{3\alpha}([- \frac{T}{3\xi_n}, \frac{T}{3\xi_n}])} & \leq C\xi_n^{-\frac{1}{2}-\frac{1}{3\alpha}} \sum_{k=2}^{\infty} \frac{|C_k|}{k^2} + C\xi_n^{-\frac{5}{4}-\frac{1}{3\alpha}} \sum_{k=2}^{\infty} \frac{|C_k|}{k^3} \\ & \leq C\xi_n^{-\frac{1}{2}-\frac{1}{3\alpha}}. \end{aligned}$$

In a similar way,

$$\|e_{n,1}\|_{L_{t,x}^{3\alpha}([- \frac{T}{3\xi_n}, \frac{T}{3\xi_n}])} \leq C\xi_n^{-\frac{3}{2}-\frac{1}{3\alpha}}.$$

Hence, we have

$$(8.20) \quad \|e_n\|_{L([- \frac{T}{3\xi_n}, \frac{T}{3\xi_n}])} \leq C\xi_n^{-\frac{3}{2}}.$$

Next we evaluate the $S([- \frac{T}{3\xi_n}, \frac{T}{3\xi_n}])$ -norm of $e_{n,1}$. We easily see

$$\begin{aligned} & \left\| (|v_n|^{2\alpha+1-k} v_n^k)(-3\xi_n t, x + 3\xi_n^2 t) \right\|_{L_x^{\frac{5\alpha}{2}} L_t^{5\alpha}([- \frac{T}{3\xi_n}, \frac{T}{3\xi_n}])} \\ & = \|v_n(-3\xi_n t, x + 3\xi_n^2 t)\|_{L_x^{\frac{5\alpha(2\alpha+1)}{2}} L_t^{5\alpha(2\alpha+1)}([- \frac{T}{3\xi_n}, \frac{T}{3\xi_n}])}^{2\alpha+1} =: I^{2\alpha+1}. \end{aligned}$$

Let us estimate I . For simplicity, we put $\rho = 5\alpha(2\alpha + 1)$. Change of variable and the Gagliardo-Nirenberg inequality yield

$$\begin{aligned} & \|v_n(-3\xi_n t, x + 3\xi_n^2 t)\|_{L_t^\rho([- \frac{T}{3\xi_n}, \frac{T}{3\xi_n}])} \\ & \leq C \xi_n^{-\frac{1}{\rho}} \|v_n(t, x - \xi_n t)\|_{L_t^\rho(\mathbb{R})} \\ & \leq C \xi_n^{-\frac{1}{\rho}} C \|v_n(t, x - \xi_n t)\|_{L_t^{\frac{\rho}{2}}(\mathbb{R})}^{1-\frac{1}{\rho}} \|\partial_t(v_n(t, x - \xi_n t))\|_{L_t^{\frac{\rho}{2}}(\mathbb{R})}^{\frac{1}{\rho}}. \end{aligned}$$

Hence,

$$I \leq C \xi_n^{-\frac{1}{\rho}} \|v_n\|_{L_{t,x}^{\frac{\rho}{2}}(\mathbb{R}^2)}^{1-\frac{1}{\rho}} \|\partial_t v_n - \xi_n \partial_x v_n\|_{L_{t,x}^{\frac{\rho}{2}}(\mathbb{R}^2)}^{\frac{1}{\rho}}.$$

Since $(\frac{\rho}{2}, \frac{2\rho}{\rho-8})$ is a Schrödinger admissible pair, it follows from (8.7) that

$$\|v_n\|_{L_{t,x}^{\frac{\rho}{2}}} \leq \left\| |\partial_x|^{\frac{1}{2}-\frac{6}{\rho}} v_n \right\|_{L_t^{\frac{\rho}{2}} L_x^{\frac{2\rho}{\rho-8}}} = O(\xi_n^{-\frac{3}{2\rho} + \frac{1}{4\alpha}}).$$

Similar estimates hold for $\partial_t v_n$ and $\partial_x v_n$. Combining above estimates, we conclude that

$$I = O(\xi_n^{\frac{1}{2(2\alpha+1)}}).$$

Thus,

$$\|e_{n,1}\|_{S([- \frac{T}{3\xi_n}, \frac{T}{3\xi_n}])} = O(\xi_n^{-\frac{3}{2}}).$$

To evaluate $e_{n,2}$, we employ the inhomogeneous estimate for Airy equation (2.5). Since $(1, \alpha)$ is a conjugate-acceptable pair,

$$\begin{aligned} (8.21) \quad & \|e_{n,2}\|_{L_t^\infty \hat{L}_x^\alpha([- \frac{T}{3\xi_n}, \frac{T}{3\xi_n}])} + \|e_{n,2}\|_{L([- \frac{T}{3\xi_n}, \frac{T}{3\xi_n}])} + \|e_{n,2}\|_{S([- \frac{T}{3\xi_n}, \frac{T}{3\xi_n}])} \\ & \leq \|R_n^1\|_{L_t^1 \hat{L}_x^\alpha([- \frac{T}{3\xi_n}, \frac{T}{3\xi_n}])} + \|R_n^2\|_{L_x^{\tilde{p}(1,\alpha)} L_t^{\tilde{q}(1,\alpha)}([- \frac{T}{3\xi_n}, \frac{T}{3\xi_n}])} \\ & \quad + \|R_n^4\|_{L_x^{\tilde{p}(1,\alpha)} L_t^{\tilde{q}(1,\alpha)}([- \frac{T}{3\xi_n}, \frac{T}{3\xi_n}])}. \end{aligned}$$

By (8.6), we have

$$(8.22) \quad \|R_n^1\|_{L_t^1 \hat{L}_x^\alpha([- \frac{T}{3\xi_n}, \frac{T}{3\xi_n}])} \leq C \xi_n^{-\frac{1}{4}} T \|v_n\|_{L_t^\infty \hat{L}_x^\alpha([-T, T])} \rightarrow 0$$

as $n \rightarrow \infty$ and

$$\begin{aligned} (8.23) \quad & \|R_n^2\|_{L_x^{\tilde{p}(1,\alpha)} L_t^{\tilde{q}(1,\alpha)}([- \frac{T}{3\xi_n}, \frac{T}{3\xi_n}])} \\ & \leq C \xi_n^{-\frac{1}{\tilde{q}(1,\alpha)}} \|v_n\|_{L_x^{2\alpha} L_t^{q(0,\alpha)}([-T, T])}^{2\alpha} \|\partial_x v_n\|_{L_x^{p(1,\alpha)} L_t^{q(1,\alpha)}([-T, T])} \\ & \leq C \xi_n^{-\frac{15}{16\tilde{q}(1,\alpha)}} \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. In a similar way

$$(8.24) \quad \|R_n^4\|_{L_x^{\tilde{p}(1,\alpha)} L_t^{\tilde{q}(1,\alpha)}([- \frac{T}{3\xi_n}, \frac{T}{3\xi_n}])} \leq C \xi_n^{-\frac{15}{16\tilde{q}(1,\alpha)}} \rightarrow 0$$

as $n \rightarrow \infty$. Combining (8.21), (8.22), (8.23) and (8.24), we have

$$(8.25) \quad \|e_n\|_{S([- \frac{T}{3\xi_n}, \frac{T}{3\xi_n}])} \rightarrow 0$$

as $n \rightarrow \infty$. From (8.19), (8.20) and (8.25), we have (8.18). \square

Lemma 8.5. (*Approximation of gKdV for middle interval*) Fix $T \in \mathbb{R}$. Let \tilde{u}_n and e_n be given by (8.10) and (8.17). Then we have

$$(8.26) \quad \lim_{n \rightarrow \infty} \|\partial_x^{-1}[(\partial_t + \partial_x^3)(\tilde{u}_n - e_n) - \mu \partial_x \{|\tilde{u}_n - e_n|^{2\alpha}(\tilde{u}_n - e_n)\}]\|_{N([- \frac{T}{3\xi_n}, \frac{T}{3\xi_n}])} = 0.$$

Proof. In the proof we omit $([- \frac{T}{3\xi_n}, \frac{T}{3\xi_n}])$, for simplicity. We first note

$$\begin{aligned} & (\partial_t + \partial_x^3)(\tilde{u}_n - e_n) - \mu \partial_x \{|\tilde{u}_n - e_n|^{2\alpha}(\tilde{u}_n - e_n)\} \\ &= \mu \partial_x \{|\tilde{u}_n|^{2\alpha} \tilde{u}_n - |\tilde{u}_n - e_n|^{2\alpha}(\tilde{u}_n - e_n)\}. \end{aligned}$$

Lemma 2.9 implies

$$\begin{aligned} & \| |\tilde{u}_n|^{2\alpha} \tilde{u}_n - |\tilde{u}_n - e_n|^{2\alpha}(\tilde{u}_n - e_n) \|_N \\ & \leq C(\|\tilde{u}_n\|_L + \|e_n\|_L)(\|\tilde{u}_n\|_S + \|e_n\|_S)^{2\alpha-1} \|e_n\|_S \\ & \quad + C(\|\tilde{u}_n\|_S + \|e_n\|_S)^{2\alpha} \|e_n\|_L. \end{aligned}$$

By Lemma 8.4, letting $n \rightarrow \infty$ in the above inequalities, we obtain (8.26). \square

Lemma 8.6 (Initial condition). Take a parameter T so that $T > T_0$ if $|T_0| < \infty$ and arbitrarily positive if $T_0 = \pm\infty$. Let $u_n(t_n)$ and $\tilde{u}_n(t)$ be given by (4.7) and (8.10), respectively. Then we have

$$(8.27) \quad \lim_{n \rightarrow \infty} \|u_n(t_n) - \tilde{u}_n(t_n)\|_{\hat{L}_x^\alpha} = 0.$$

Proof. We first consider the case $|T_0| < \infty$. Notice that in this case we necessarily have $t_n \rightarrow 0$ as $n \rightarrow \infty$. Since $|t_n| \leq T/(3\xi_n)$ for n sufficiently large, we have

$$\begin{aligned} & \|u_n(t_n) - \tilde{u}_n(t_n)\|_{\hat{L}_x^\alpha} \\ & \leq \|e^{-it_n \partial_x^3} [e^{-ix\xi_n} \phi(x)] - e^{-ix\xi_n - it_n \xi_n^3} v_n(-3\xi_n t_n, x + 3\xi_n^2 t_n)\|_{\hat{L}_x^\alpha} \\ & = \|e^{it_n(\xi - \xi_n)^3} \hat{\phi}(\xi) - e^{-it_n \xi_n^3 + 3it_n \xi_n^2 \xi} \hat{v}_n(-3\xi_n t_n, \xi)\|_{L_\xi^{\alpha'}} \\ & = \|e^{it_n \xi^3 - 3it_n \xi_n \xi^2} \hat{\phi}(\xi) - \hat{v}_n(-3\xi_n t_n, \xi)\|_{L_\xi^{\alpha'}}. \end{aligned}$$

Since $t_n \rightarrow 0$ and $-3t_n \xi_n \rightarrow T_0$, we have

$$\begin{aligned} & \|e^{it_n \xi^3 - 3it_n \xi_n \xi^2} \hat{\phi}(\xi) - \hat{v}_n(-3\xi_n t_n, \xi)\|_{L_\xi^{\alpha'}} \\ & \leq \|(e^{it_n \xi^3} - 1)e^{-3it_n \xi_n \xi^2} \hat{\phi}(\xi)\|_{L_\xi^{\alpha'}} + \|e^{-3it_n \xi_n \xi^2} \hat{\phi}(\xi) - \hat{v}(-3\xi_n t_n, \xi)\|_{L_\xi^{\alpha'}} \\ & \quad + \|\hat{v}(-3\xi_n t_n, \xi) - \hat{v}_n(-3\xi_n t_n, \xi)\|_{L_\xi^{\alpha'}} \\ & \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$, where we used the dominated convergence theorem, (8.2) and (8.5). This proves (8.27) for $|T_0| < \infty$.

Next we consider the case $T_0 = \pm\infty$. We treat the case $T_0 = \infty$ only since the case $T_0 = -\infty$ being similar. For n sufficiently large, $-3t_n \xi_n > T$. Hence

$$\|u_n(t_n) - \tilde{u}_n(t_n)\|_{\hat{L}_x^\alpha}$$

$$\begin{aligned}
&\leq \|e^{-(t_n + \frac{T}{3\xi_n})\partial_x^3} [e^{\frac{T}{3\xi_n}\partial_x^3} [e^{-ix\xi_n}\phi(x)] - e^{-ix\xi_n + \frac{i}{3}T\xi_n^2} v_n(T, x - \xi_n T)]\|_{\hat{L}_x^\alpha} \\
&= \|e^{\frac{T}{3\xi_n}\partial_x^3} [e^{-ix\xi_n}\phi(x)] - e^{-ix\xi_n + \frac{i}{3}T\xi_n^2} v_n(T, x - \xi_n T)\|_{\hat{L}_x^\alpha} \\
&= \|e^{-i\frac{T}{3\xi_n}(\xi - \xi_n)^3} \hat{\phi}(\xi) - e^{\frac{i}{3}T\xi_n^2 - iT\xi_n\xi} \hat{v}_n(T, \xi)\|_{L_\xi^{\alpha'}} \\
&= \|e^{-i\frac{T}{3\xi_n}\xi^3 + iT\xi^2} \hat{\phi}(\xi) - \hat{v}_n(T, \xi)\|_{L_\xi^{\alpha'}}.
\end{aligned}$$

Since $\xi_n \rightarrow \infty$, the dominated convergence theorem, (8.2) and (8.5) yield

$$\begin{aligned}
&\|e^{-i\frac{T}{3\xi_n}\xi^3 + iT\xi^2} \hat{\phi}(\xi) - \hat{v}_n(T, \xi)\|_{L_\xi^{\alpha'}} \\
&\leq \|(e^{-i\frac{T}{3\xi_n}\xi^3} - 1)e^{iT\xi^2} \hat{\phi}(\xi)\|_{L_\xi^{\alpha'}} + \|e^{iT\xi^2} \hat{\phi}(\xi) - \hat{v}(T, \xi)\|_{L_\xi^{\alpha'}} \\
&\quad + \|\hat{v}(T, \xi) - \hat{v}_n(T, \xi)\|_{L_\xi^{\alpha'}} \\
&\rightarrow 0
\end{aligned}$$

as $n \rightarrow \infty$, which proves (8.27) for the case $T_0 = \infty$. This completes the proof of Lemma 8.6. \square

Proof of Theorem 4.4. By Lemma 8.1, there exist two positive constants A and M which are independent of T and n such that

$$\begin{aligned}
\|\tilde{u}_n\|_{L_t^\infty(\mathbb{R}; \hat{L}_x^\alpha)} &\leq A, \\
\|\tilde{u}_n\|_{S(\mathbb{R})} + \|\tilde{u}_n\|_{L(\mathbb{R})} &\leq M.
\end{aligned}$$

For the above M , let $\varepsilon_1 = \varepsilon_1(M)$ be given by Lemma 3.2 and let C be a constant appearing in Lemma 3.2. Then, Lemma 8.3 yields that for any ε satisfying $0 < \varepsilon < C\varepsilon_1$, there exists a positive constant T_ε such that if $T \geq T_\varepsilon$, then

(8.28)

$$\lim_{n \rightarrow \infty} \| |\partial_x|^{-1} \{ (\partial_t + \partial_x^3) \tilde{u}_n - \mu \partial_x (|\tilde{u}_n|^{2\alpha} \tilde{u}_n) \} \|_{N(|t| > \frac{T}{3\xi_n})} < \frac{\varepsilon}{2}.$$

We now choose

$$T := \begin{cases} \max\{T_\varepsilon, 2|T_0|\} & \text{if } T_0 = \pm\infty, \\ T_\varepsilon & \text{if } |T_0| < \infty. \end{cases}$$

We first apply the long time stability for gKdV in the time interval $\{|t| \leq T/(3\xi_n)\}$. Lemmas 8.5 and 8.6 lead that there exists a nonnegative integer $N_1 = N_1(\varepsilon, T_\varepsilon)$ such that if $n \geq N_1$, then $|t_n| \leq T/(3\xi_n)$ and

$$\begin{aligned}
&\|u_n(t_n) - \tilde{u}_n(t_n)\|_{\hat{L}_x^\alpha} \\
&\quad + \| |\partial_x|^{-1} \{ (\partial_t + \partial_x^3) \tilde{u}_n - \mu \partial_x (|\tilde{u}_n|^{2\alpha} \tilde{u}_n) \} \|_{N(|t| \leq \frac{T}{3\xi_n})} \leq \frac{\varepsilon}{2}.
\end{aligned}$$

Hence, by Proposition 3.2, there exists a unique solution $u \in C(I; \hat{L}_x^\alpha)$ to (gKdV) satisfying

$$(8.29) \quad \|u_n - \tilde{u}_n\|_{L_t^\infty(I; \hat{L}_x^\alpha)} + \|u_n - \tilde{u}_n\|_{S(I)} + \|u_n - \tilde{u}_n\|_{L(I)} \leq \frac{\varepsilon}{2},$$

where $I = [-\frac{T}{3\xi_n}, \frac{T}{3\xi_n}]$. Especially, we have

$$(8.30) \quad \left\| u_n \left(\pm \frac{T}{3\xi_n} \right) - \tilde{u}_n \left(\pm \frac{T}{3\xi_n} \right) \right\|_{\hat{L}_x^\alpha} \leq \frac{\varepsilon}{2}.$$

Next we apply the long time stability for gKdV in the time intervals $t \geq T/(3\xi_n)$ and $t \leq -T/(3\xi_n)$, respectively. Combining (8.28), (8.29), (8.30) and Lemma 3.2, we find that there exists a unique global solution $u \in C(\mathbb{R}; \hat{L}_x^\alpha)$ to (gKdV) satisfying

$$\|u_n - \tilde{u}_n\|_{L_t^\infty(\mathbb{R}; \hat{L}_x^\alpha)} + \|u_n - \tilde{u}_n\|_S + \|u_n - \tilde{u}_n\|_L \leq C\varepsilon.$$

Combining the above inequality and Lemma 8.1 we have Theorem 4.4.

APPENDIX A. ON GENERALIZED MORREY SPACES

In this appendix, we give the following interpolation type inequality for the generalized Morrey spaces.

Proposition A.1. *Suppose that $0 < q < p < r < \infty$. If s satisfies*

$$\frac{1}{s} \times \left(1 - \frac{p}{r}\right) + \frac{1}{p} \times \frac{p}{r} < \frac{1}{q}$$

then, for any $f \in L^q(\mathbb{R})$, we have

$$\|f\|_{M_{q,r}^p} \leq C \|f\|_{M_{s,\infty}^p}^{1-\frac{p}{r}} \|f\|_{M_{p,\infty}^p}^{\frac{p}{r}}.$$

In particular, $L^p \hookrightarrow M_{q,r}^p$.

Proof. Set

$$f_{n,I}(x) := f(x) \mathbf{1}_{I \cap \{2^n \leq |I|^{\frac{1}{p}} |f(x)| \leq 2^{n+1}\}}(x)$$

for $I \in \mathcal{D}$ and $n \in \mathbb{Z}$. Let $\theta = 1 - p/r$. By the Hölder inequality in x ,

$$\begin{aligned} (A.1) \quad \int |f_{n,I}(x)|^q dx &= \int |f_{n,I}(x)|^{\theta q} |f_{n,I}(x)|^{(1-\theta)q} dx \\ &\leq \left(\int |f_{n,I}(x)|^{\frac{\theta q p}{p-(1-\theta)q}} dx \right)^{1-\frac{(1-\theta)q}{p}} \left(\int |f_{n,I}(x)|^p dx \right)^{\frac{(1-\theta)q}{p}}. \end{aligned}$$

By definition of $f_{n,I}$, we have

$$\left(\int |f_{n,I}(x)|^{\frac{\theta q p}{p-(1-\theta)q}} dx \right)^{1-\frac{(1-\theta)q}{p}} \leq C 2^{\theta q n} |I|^{-\frac{\theta q}{p}} \left(\int_{I \cap \{|f| \geq 2^n |I|^{-\frac{1}{p}}\}} dx \right)^{1-\frac{(1-\theta)q}{p}}.$$

It is obvious that

$$\left(\int_{I \cap \{|f| \geq 2^n |I|^{-\frac{1}{p}}\}} dx \right)^{1-\frac{(1-\theta)q}{p}} \leq |I|^{1-\frac{(1-\theta)q}{p}}.$$

One sees from Chebyshev's inequality that

$$\left(\int_{I \cap \{|f| \geq 2^n |I|^{-\frac{1}{p}}\}} dx \right)^{1-\frac{(1-\theta)q}{p}}$$

$$\begin{aligned}
&\leq \left(\frac{\int_I |f(x)|^s dx}{2^{ns} |I|^{-\frac{s}{p}}} \right)^{1 - \frac{(1-\theta)q}{p}} \\
&\leq 2^{-(1 - \frac{(1-\theta)q}{p})sn} |I|^{1 - \frac{(1-\theta)q}{p}} \left(\sup_{I \in \mathcal{D}} |I|^{\frac{1}{p} - \frac{1}{s}} \|f\|_{L^s(I)} \right)^{(1 - \frac{(1-\theta)q}{p})s}.
\end{aligned}$$

Namely,

$$\begin{aligned}
&\left(\int |f_{n,I}(x)|^{\frac{\theta pq}{p - (1-\theta)q}} dx \right)^{1 - \frac{(1-\theta)q}{p}} \\
&\leq C |I|^{1 - \frac{q}{p}} \min \left(2^{\theta q n}, 2^{\theta q n - (1 - \frac{(1-\theta)q}{p})sn} \|f\|_{M_{s,\infty}^p}^{(1 - \frac{(1-\theta)q}{p})s} \right) \\
&= C |I|^{1 - \frac{q}{p}} \|f\|_{M_{s,\infty}^p}^{\theta q} \min \left(2^{\theta q(n-n_0)}, 2^{(\theta - \frac{s}{q} + \frac{(1-\theta)s}{p})q(n-n_0)} \right),
\end{aligned}$$

where, we chose $n_0 \in \mathbb{R}$ by $2^{n_0} = \|f\|_{M_{s,\infty}^q}$. Since

$$\theta - \frac{s}{q} + \frac{(1-\theta)s}{p} < 0 < \theta$$

by assumption, there exists $\delta = \delta(p, q, s, \theta) > 0$ such that

$$(A.2) \quad \left(\int |f_{n,I}(x)|^{\frac{\theta pq}{q - (1-\theta)p}} dx \right)^{1 - \frac{(1-\theta)p}{q}} \leq C 2^{-\delta|n-n_0|} |I|^{1 - \frac{q}{p}} \|f\|_{M_{s,\infty}^p}^{\theta q}$$

for all $n \in \mathbb{Z}$ and $I \in \mathcal{D}$.

Note that

$$\int_I |f(x)|^q dx = \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} |f_{n,I}(x)|^q dx$$

for any $I \in \mathcal{D}$ since $q < \infty$ by assumption. The inequalities (A.1) and (A.2) yield

$$\begin{aligned}
&\|f\|_{M_{q,r}^p}^r \\
&= \sum_{I \in \mathcal{D}} \left(\sum_{n \in \mathbb{Z}} |I|^{\frac{q}{p} - 1} \|f_{n,I}\|_{L^q(\mathbb{R})}^q \right)^{r/q} \\
&\leq C \sum_{I \in \mathcal{D}} \left(\sum_{n \in \mathbb{Z}} 2^{-\delta|n-n_0|} \|f\|_{M_{s,\infty}^p}^{\theta q} \left(\int |f_{n,I}(x)|^p dx \right)^{\frac{(1-\theta)q}{p}} \right)^{r/q} \\
&= C \|f\|_{M_{s,\infty}^p}^{\theta r} \sum_{I \in \mathcal{D}} \left(\sum_{n \in \mathbb{Z}} \left(2^{-\delta'|n-n_0|} \int |f_{n,I}(x)|^p dx \right)^{q/r} \right)^{r/q} \\
&\leq C_{\delta'} \|f\|_{M_{s,\infty}^p}^{\theta r} \sum_{I \in \mathcal{D}} \sum_{n \in \mathbb{Z}} 2^{-\frac{\delta'}{2}|n-n_0|} \int |f_{n,I}(x)|^p dx,
\end{aligned}$$

where we have used the Hölder inequality in n to yield the last line. Thus,

$$\|f\|_{M_{q,r}^p}^r \leq C \|f\|_{M_{s,\infty}^p}^{\theta r} \sup_{n \in \mathbb{Z}} \sum_{I \in \mathcal{D}} \int |f_{n,I}(x)|^p dx$$

Finally, for any fixed n , we have

$$\begin{aligned} \sum_{I \in \mathcal{D}} \int |f_{n,I}(x)|^p dx &= \sum_{j \in \mathbb{Z}} \sum_{I \in \mathcal{D}_j} \int |f_{n,I}(x)|^p dx \\ &= \sum_{j \in \mathbb{Z}} \int_{\{2^n \leq 2^{-\frac{j}{p}} |f(x)| \leq 2^{n+1}\}} |f(x)|^p dx, \end{aligned}$$

where we have used the fact that elements of \mathcal{D}_j are mutually disjoint and $\cup_{I \in \mathcal{D}_j} I = \mathbb{R}$. Since $\{2^n \leq 2^{-\frac{j}{p}} |f(x)| \leq 2^{n+1}\}$ and $\{2^n \leq 2^{-\frac{j'}{p}} |f(x)| \leq 2^{n+1}\}$ are disjoint as long as $|j - j'| > p$, we have

$$\sum_{j \in \mathbb{Z}} \int_{\{2^n \leq 2^{-\frac{j}{p}} |f(x)| \leq 2^{n+1}\}} |f(x)|^p dx \leq (p+1) \|f\|_{L^p(\mathbb{R})}^p = (p+1) \|f\|_{L^p(\mathbb{R})}^{(1-\theta)r},$$

which completes the proof. \square

APPENDIX B. ON REFINED STEIN-TOMAS ESTIMATE

In this subsection, we prove the first inequality of the refined Stein-Tomas estimate (6.4).

Theorem B.1. *Let $4/3 \leq p < \infty$. Then, there exist a constant $C = C(p)$ such that*

$$(B.1) \quad \left\| |\partial_x|^{1/3p} e^{-t\partial_x^3} f \right\|_{L_{t,x}^{3p}} \leq C \|f\|_{\hat{M}_{\frac{3p}{2}, 2(\frac{3p}{2})'}^p}.$$

for any $f \in \hat{M}_{\frac{3p}{2}, 2(\frac{3p}{2})'}^p$.

Proof. We argue as in Shao [51]. The square of the left hand side of (B.1) is equal to

$$\left\| \iint e^{ix(\xi-\eta) + it(\xi^3 - \eta^3)} |\xi\eta|^{1/3p} \hat{f}(\xi) \overline{\hat{f}(\eta)} d\xi d\eta \right\|_{L_{t,x}^{3p/2}}.$$

Changing variables by $a = \xi - \eta$ and $b = \xi^3 - \eta^3$, we have

$$\left\| \iint e^{ixa + itb} |\xi\eta|^{\frac{1}{3p}} \hat{f}(\xi) \overline{\hat{f}(\eta)} \frac{1}{3|\xi^2 - \eta^2|} da db \right\|_{L_{t,x}^{3p/2}}.$$

Since $3p/2 \geq 2$, we use Hausdorff-Young inequality to deduce that this is bounded by

$$\begin{aligned} & C \left\| |\xi\eta|^{1/3p} \hat{f}(\xi) \overline{\hat{f}(\eta)} |\xi^2 - \eta^2|^{-1} \right\|_{L_{a,b}^{(3p/2)'}} \\ &= C \left\{ \iint_{\mathbb{R}^2} \frac{|\xi\eta|^{\frac{1}{3p-2}} |\hat{f}(\xi)|^{(\frac{3p}{2})'} |\hat{f}(\eta)|^{(\frac{3p}{2})'}}{|\xi - \eta|^{\frac{2}{3p-2}} |\xi + \eta|^{\frac{2}{3p-2}}} d\xi d\eta \right\}^{1 - \frac{2}{3p}}. \end{aligned}$$

Thus, we have

$$\left\| |\partial_x|^{1/3p} e^{-t\partial_x^3} f \right\|_{L_{t,x}^{3p}}^{2(\frac{3p}{2})'} \leq C \iint_{\mathbb{R}^2} \frac{|\xi\eta|^{\frac{1}{3p-2}} |\hat{f}(\xi)|^{(\frac{3p}{2})'} |\hat{f}(\eta)|^{(\frac{3p}{2})'}}{|\xi + \eta|^{\frac{2}{3p-2}} |\xi - \eta|^{\frac{2}{3p-2}}} d\xi d\eta.$$

We now introduce a Whitney-type decomposition. For an interval $I \in \mathcal{D}_j$, there exists a unique interval $J \in \mathcal{D}_{j-1}$ such that $I \subset J$. We call J as a

parent of I . For two intervals $I, I' \in \mathcal{D}$, we introduce a binary relation $\sim_{\mathcal{W}}$ so that $I \sim_{\mathcal{W}} I'$ holds if the following three conditions are satisfied; (i) I and I' belong to same \mathcal{D}_j , that is, $|I| = |I'|$; (ii) I is not neighboring neither I' nor $-I'$; and (iii) a parent of I is neighboring either a parent of I' or a parent of $-I'$. Set $\mathcal{W} := \{(I, I') \in \mathcal{D} \times \mathcal{D} \mid I \sim_{\mathcal{W}} I'\}$. Notice that if $I \sim_{\mathcal{W}} I'$ then $|I| \leq \min(\text{dist}(I, I'), \text{dist}(I, -I')) \leq 2|I|$ and that for any $I \in \mathcal{D}$, $\#\{I' \in \mathcal{D} \mid I \sim_{\mathcal{W}} I'\} = 2, 4$ or 6 . Then, we have the following Whitney-type decomposition of $\mathbb{R} \times \mathbb{R}$;

$$\sum_{(I, I') \in \mathcal{W}} \mathbf{1}_I(\xi) \mathbf{1}_{I'}(\eta) = 1, \quad (\xi, \eta) \in \mathbb{R}^2 \setminus \{(\xi, \pm\xi) \mid \xi \in \mathbb{R}\}.$$

Let \mathcal{W} be as above. Since $|\xi\eta| \leq \max(|\xi + \eta|^2, |\xi - \eta|^2)$ for any $(\xi, \eta) \in \mathbb{R}^2$, one sees that

$$\frac{|\xi\eta|}{|\xi + \eta|^2 |\xi - \eta|^2} \leq |I|^{-2}$$

for any $(\xi, \eta) \in I \times I'$ with $(I, I') \in \mathcal{W}$. We hence obtain

$$\begin{aligned} & \iint_{\mathbb{R}^2} \frac{|\xi\eta|^{\frac{1}{3p-2}} |\hat{f}(\xi)|^{(\frac{3p}{2})'} |\hat{f}(\eta)|^{(\frac{3p}{2})'}}{|\xi + \eta|^{\frac{2}{3p-2}} |\xi - \eta|^{\frac{2}{3p-2}}} d\xi d\eta \\ &= \iint_{\mathbb{R}^2} \sum_{(I, I') \in \mathcal{W}} \left(\frac{|\xi\eta|}{|\xi + \eta|^2 |\xi - \eta|^2} \right)^{\frac{1}{3p-2}} |\hat{f}(\xi)|^{(\frac{3p}{2})'} |\hat{f}(\eta)|^{(\frac{3p}{2})'} \mathbf{1}_I(\xi) \mathbf{1}_{I'}(\eta) d\xi d\eta \\ &\leq \sum_{I \in \mathcal{D}} \sum_{I' \in \mathcal{W} I'} |I|^{-\frac{2}{3p-2}} \int_I |\hat{f}(\xi)|^{(\frac{3p}{2})'} d\xi \int_{I'} |\hat{f}(\eta)|^{(\frac{3p}{2})'} d\eta. \end{aligned}$$

We choose a slightly larger interval containing I and either I' or $-I'$ but still of length comparable to I^4 , still denote by I , we obtain

$$\begin{aligned} & \iint_{\mathbb{R}^2} \frac{|\xi\eta|^{\frac{1}{3p-2}} |\hat{f}(\xi)|^{(\frac{3p}{2})'} |\hat{f}(\eta)|^{(\frac{3p}{2})'}}{|\xi + \eta|^{\frac{2}{3p-2}} |\xi - \eta|^{\frac{2}{3p-2}}} d\xi d\eta \\ &\leq C \sum_{I \in \mathcal{D}} |I|^{-\frac{2}{3p-2}} \left(\int_{I \cup (-I)} |\hat{f}(\xi)|^{(\frac{3p}{2})'} d\xi \right)^2 \\ &\leq C \sum_{I \in \mathcal{D}} |I|^{-\frac{2}{3p-2}} \|\hat{f}\|_{L^{(\frac{3p}{2})'}(I)}^{\frac{6p}{3p-2}} \\ &= C \|f\|_{\dot{M}_{\frac{3p}{2}, 2(\frac{3p}{2})'}}^{2(\frac{3p}{2})'}, \end{aligned}$$

which completes the proof. \square

Remark B.2. Proposition 7.5 can be shown in the same way (see also [1]). More precisely, we have

$$\left\| e^{-it\partial_x^2} f \right\|_{L_{t,x}^{3p}}^{2(\frac{3p}{2})'} \leq C \iint_{\mathbb{R}^2} \frac{|\hat{f}(\xi)|^{(\frac{3p}{2})'} |\hat{f}(\eta)|^{(\frac{3p}{2})'}}{|\xi - \eta|^{\frac{2}{3p-2}}} d\xi d\eta.$$

The rest of proof is essentially the same.

⁴More specifically, it is enough to take a parent of a parent of a parent of I .

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REFERENCES

- [1] Bégout P. and Vargas A., *Mass concentration phenomena for the L^2 -critical nonlinear Schrödinger equation*. Trans. Amer. Math. Soc. **359** (2007), 5257–5282.
- [2] Benjamin T.B. *The stability of solitary waves*. Proc. Roy. Soc. (London) Ser. A **328** (1972), 153–183.
- [3] Bona J.L., Souganidis P.E. and Strauss W.A., *Stability and instability of solitary waves of Korteweg-de Vries type*. Proc. Roy. Soc. London Ser. A **411** (1987), 395–412.
- [4] Bourgain J., *On the restriction and multiplier problems in \mathbb{R}^3* . Geometric aspects of functional analysis (1989–90), Lecture Notes in Math., **1469**, Springer, Berlin, (1991), 179–191.
- [5] Bourgain J., *Some new estimates on oscillatory integrals*. Essays on Fourier analysis in honor of Elias M. Stein (Princeton, NJ, 1991), Princeton Math. Ser., 42, Princeton Univ. Press (1995), 83–112.
- [6] Bourgain J., *Refinements of Strichartz’ inequality and applications to 2D-NLS with critical nonlinearity*. Internat. Math. Res. Notices **1998** (1998), 253–283.
- [7] Boyd J.P. and Chen G., *Weakly nonlinear wavepackets in the Korteweg-de Vries equation: the KdV/NLS connection*. Math. Comput. Simulation **55** (2001), 317–328.
- [8] Carles R., and Keraani S., *On the role of quadratic oscillations in nonlinear Schrödinger equations II. The L^2 -critical case*. Trans. Amer. Math. Soc. **359** (2007), 33–62.
- [9] Cazenave T., “*Semilinear Schrödinger equations*”. Courant Lecture Notes in Mathematics, **10**. American Mathematical Society (2003).
- [10] Christ M., Colliander J. and Tao T., *Asymptotics, frequency modulation, and low regularity ill-posedness for canonical defocusing equations*. Amer. J. Math. **125** (2003), 1235–1293.
- [11] Christ F.M. and Weinstein M.I., *Dispersion of small amplitude solutions of the generalized Korteweg-de Vries equation*. J. Funct. Anal. **100** (1991) 87–109.
- [12] Deift P. and Zhou X., *A steepest descent method for oscillatory Riemann-Hilbert problems. Asymptotics for the MKdV equation*. Ann. of Math. (2) **137** (1993), 295–368.
- [13] Dodson. B., *Global well-posedness and scattering for the defocusing, L^2 -critical, nonlinear Schrödinger equation when $d = 1$* . To appear in Amer. J. Math. preprint available at arXiv:1010.0040.
- [14] Dodson. B., *Global well-posedness and scattering for the defocusing, mass-critical generalized KdV equation*. preprint available at arXiv:1304.8025 (2013).
- [15] Grünrock A., *An improved local well-posedness result for the modified KdV equation*. Int. Math. Res. Not. **2004** (2004), 3287–3308.
- [16] Grünrock A., *Bi- and trilinear Schrödinger estimates in one space dimension with applications to cubic NLS and DNLS*. Int. Math. Res. Not. **2005**, (2005), 2525–2558.
- [17] Grünrock A. and Vega L., *Local well-posedness for the modified KdV equation in almost critical \dot{H}^r -spaces*. Trans. Amer. Math. Soc. **361** (2009), 5681–5694.
- [18] Hayashi N. and Naumkin P.I., *Large time asymptotics of solutions to the generalized Korteweg-de Vries equation*. J. Funct. Anal. **159** (1998) 110–136.
- [19] Hayashi N. and Naumkin P.I., *Large time behavior of solutions for the modified Korteweg-de Vries equation*. Internat. Math. Res. Notices **1999** (1999), 395–418.
- [20] Hayashi N. and Naumkin P.I., *On the modified Korteweg-de Vries equation*. Math. Phys. Anal. Geom. **4** (2001), 197–227.

- [21] Hayashi N. and Naumkin P.I., *Final state problem for Korteweg-de Vries type equations*. J. Math. Phys. **47** (2006), 16 pp.
- [22] Hyakuna R. and Tsutsumi M., *On existence of global solutions of Schrödinger equations with subcritical nonlinearity for \hat{L}^p -initial data*. Proc. Amer. Math. Soc. **140** (2012), 3905–3920.
- [23] Kato T., *On the Cauchy problem for the (generalized) KdV equation*. Advances in Math. Supplementary studies, Studies in Applied Mathematics **8** (1983), 93–128.
- [24] Kato T., *An $L^{q,r}$ -theory for nonlinear Schrödinger equations*. Spectral and scattering theory and applications, Adv. Stud. Pure Math., vol. 23, Math. Soc. Japan, Tokyo, (1994), 223–238.
- [25] Kenig C.E. and Merle F., *Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case*. Invent. Math. **166** (2006), 645–675.
- [26] Kenig C.E., Ponce G. and Vega L., *Oscillatory integrals and regularity of dispersive equations*. Indiana Univ. math J. **40** (1991), 33–69.
- [27] Kenig C.E., Ponce G. and Vega L., *Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle*. Comm. Pure Appl. Math. **46** (1993), 527–620.
- [28] Kenig C.E., Ponce G. and Vega L., *On the concentration of blow up solutions for the generalized KdV equation critical in L^2* . Nonlinear wave equations (Providence, RI 1998), Contemp. Math. **263**, Amer. Math. Soc., Providence, RI (2000), 131–156.
- [29] Keraani S., *On the defect of compactness for the Strichartz estimates of the Schrödinger equations*. J. Differential Equations **175** (2001), 353–392.
- [30] Killip R, Kwon S., Shao S. and Visan M., *On the mass-critical generalized KdV equation*. Discrete Contin. Dyn. Syst. **32** (2012), 191–221.
- [31] Koch H. and Marzuola J.L., *Small data scattering and soliton stability in $\dot{H}^{-\frac{1}{6}}$ for the quartic KdV equation*. Anal. PDE **5** (2012), 145–198.
- [32] Korteweg D. J. and de Vries G., *On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves*, Philos. Mag. **39** (1895), 422–443.
- [33] Lamb G.L.Jr., *Solitons on moving space curves*. J. Math. Phys. **18** (1977), 1654–1661.
- [34] Martel Y. and Merle F., *Instability of solitons for the critical generalized Korteweg-de Vries equation*. Geom. Funct. Anal. **11** (2001), 74–123.
- [35] Martel Y. and Merle F., *Blow up in finite time and dynamics of blow up solutions for the critical generalized KdV equation*. J. Amer. Math. Soc. **15** (2002), 617–664.
- [36] Martel Y., Merle F., Nakanishi K. and Raphaël P., *Codimension one threshold manifold for the critical gKdV equation*. To appear in Comm. Math. Phys. preprint available at arXiv:1502.04594.
- [37] Martel Y., Merle F. and Raphaël P., *Blow up for the critical generalized Korteweg de Vries equation. I: Dynamics near the soliton*. Acta Math. **212** (2014), 59–140.
- [38] Martel Y., Merle F. and Raphaël P., *Blow up for the critical gKdV equation. II: Minimal mass dynamics*. J. Eur. Math. Soc. (JEMS) **17** (2015), 1855–1925.
- [39] Martel Y., Merle F. and Raphaël P., *Blow up for the critical gKdV equation III : exotic regimes*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **14** (2015), 575–631.
- [40] Masaki S., *On minimal non-scattering solution for focusing mass-subcritical nonlinear Schrödinger equation*. preprint available at arXiv:1301.1742 (2013).
- [41] Masaki S., *A sharp scattering condition for focusing mass-subcritical nonlinear Schrödinger equation*. Commun. Pure Appl. Anal. **14** (2015), 1481–1531.
- [42] Masaki S. and Segata J., *On well-posedness of generalized Korteweg-de Vries equation in scale critical \hat{L}^r space*. To appear in Anal. PDE. preprint available at arXiv:1507.01323.
- [43] Merle F. and Vega L., *Compactness at blow-up time for L^2 solutions of the critical nonlinear Schrödinger equation in 2D*. Internat. Math. Res. Notices **1998** (1998), 399–425.
- [44] Molinet L. and Ribaud F., *Well-posedness results for the generalized Benjamin-Ono equation with small initial data*. J. Math. Pures Appl. **83** (2004), 277–311.

- [45] Moyua, A., Vargas, A., and Vega, L., *Schrödinger maximal function and restriction properties of the Fourier transform*. Internat. Math. Res. Notices **1996** (1996), 793–815.
- [46] Moyua, A., Vargas, A., and Vega, L., *Restriction theorems and maximal operators related to oscillatory integrals in \mathbb{R}^3* . Duke Math. J. **96** (1999), no. 3, 547–574.
- [47] Ponce G. and Vega L., *Nonlinear small data scattering for the generalized Korteweg-de Vries equation*. J. Funct. Anal. **90** (1990), 445–457.
- [48] Rammaha M.A., *On the asymptotic behavior of solutions of generalized Korteweg-de Vries equations*. J. Math. Anal. Appl. **140** (1989), 228–240.
- [49] Schindler, I. and Tintarev, K., *An abstract version of the concentration compactness principle*. Rev. Mat. Complut. **15** (2002), 417–436.
- [50] Schneider G., *Approximation of the Korteweg-de Vries equation by the nonlinear Schrödinger equation*. J. Differential Equations **147** (1998), 333–354.
- [51] Shao S., *The linear profile decomposition for the Airy equation and the existence of maximizers for the Airy Strichartz inequality*. Anal. PDE **2** (2009), 83–117.
- [52] Sidi A., Sulem C., and Sulem P. L., *On the long time behavior of a generalized KdV equation*. Acta Applicandae Math. **7** (1986), 35–47.
- [53] Strauss W.A., *Nonlinear scattering theory at low energy*. J. Funct. Anal. **41** (1981), 110–133.
- [54] Tao T., *Scattering for the quartic generalised Korteweg-de Vries equation*. J. Differential Equations **232** (2007), 623–651.
- [55] Tao T., *Two remarks on the generalised Korteweg-de Vries equation*. Discrete Contin. Dyn. Syst. **18** (2007), 1–14.
- [56] Tomas P. A., *A restriction theorem for the Fourier transform*. Bull. Amer. Math. Soc. **81** (1975), 477–478.
- [57] Vargas A. and Vega L., *Global wellposedness for 1D non-linear Schrödinger equation for data with an infinite L^2 norm*. J. Math. Pures Appl. (9) **80** (2001), 1029–1044.
- [58] Weinstein M.I., *Nonlinear Schrödinger equations and sharp interpolation estimates*. Comm. Math. Phys. **87** (1982/83), 567–576.
- [59] Weinstein M.I., *Lyapunov stability of ground states of nonlinear dispersive evolution equations*. Comm. Pure Appl. Math. **39** (1986), 51–67.

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